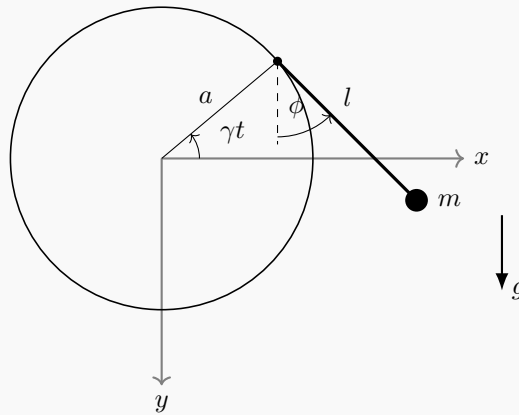


PROBLEM 3.

Find the Lagrangian of a simple pendulum of mass m , when placed in a uniform gravitational field (acceleration g), whose point of support ...



(a)

moves uniformly on a vertical circle with constant frequency γ .

SOLUTION: If we set that the rotation of the point of support is counterclockwise, the Cartesian position of the particle m is

$$\begin{aligned} x &= f(\phi_1) = a \cos \gamma t + l \sin \phi \\ y &= g(\phi_1) = -a \sin \gamma t + l \cos \phi. \end{aligned}$$

The time derivative of the position is then

$$\begin{aligned} \dot{x} &= -a\gamma \sin \gamma t + l\dot{\phi} \cos \phi \\ \dot{y} &= -a\gamma \cos \gamma t - l\dot{\phi} \sin \phi. \end{aligned}$$

The potential energy of the system is

$$\begin{aligned} U &= -mgy \\ &= -mg(-a \sin \gamma t + l \cos \phi). \end{aligned}$$

The term $mg a \sin \gamma t$ only depend on time and can therefore be ignored (does not contribute to the equations of motion). The kinetic energy of the particle, for its part, is

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2}m \left(l^2 \dot{\phi}^2 + a^2 \gamma^2 + 2a\gamma l \dot{\phi} \sin(\phi - \gamma t) \right), \end{aligned}$$

using again the angle difference identity (I will stop mentioning it). We can observe that the term $\frac{1}{2}ma^2\gamma^2$ is a constant, thus can be ignored. The last term of the kinetic energy can also be simplified. Indeed,

$$\begin{aligned} ma\gamma l \dot{\phi} \sin(\phi - \gamma t) &= mal\gamma(\dot{\phi} - \gamma) \sin(\phi - \gamma t) + mal\gamma^2 \sin(\gamma t - \phi) \\ &= \frac{d}{dt} (-mal\gamma \cos(\phi - \gamma t)) + mal\gamma^2 \sin(\phi - \gamma t). \end{aligned}$$

After dropping the total time derivative, we can finally get the Lagrangian

ANSWER TO PROBLEM 3 (A).

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2 \sin(\phi - \gamma t) + mgl \cos \phi$$

(b)

oscillates horizontally in the plane of motion of the pendulum according to the law $x = a \cos \gamma t$.

SOLUTION: The Cartesian position of the particle m is

$$\begin{aligned} x &= f(\phi_1) = a \cos \gamma t + l \sin \phi \\ y &= g(\phi_1) = l \cos \phi. \end{aligned}$$

The time derivative of the position is then

$$\begin{aligned} \dot{x} &= -a\gamma \sin \gamma t + l\dot{\phi} \cos \phi \\ \dot{y} &= -l\dot{\phi} \sin \phi. \end{aligned}$$

The potential energy of the system is

$$U = -mgy = -mgl \cos \phi,$$

and the kinetic energy

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2}m \left(l^2 \dot{\phi}^2 + a^2 \gamma^2 \sin^2 \gamma t - 2a\gamma l \dot{\phi} \sin \gamma t \cos \phi \right). \end{aligned}$$

We first see that the term $\frac{1}{2}ma^2\gamma^2 \sin^2 \gamma t$ only depend on time. The last term of the kinetic energy can also be simplified. Indeed,

$$\begin{aligned} -ma\gamma l \dot{\phi} \sin \gamma t \cos \phi &= -mla\gamma (\gamma \cos \gamma t \sin \phi + \dot{\phi} \sin \gamma t \cos \phi) + mla\gamma^2 \cos \gamma t \sin \phi \\ &= \frac{d}{dt} (-mla\gamma \sin \gamma t \sin \phi) + mla\gamma^2 \cos \gamma t \sin \phi. \end{aligned}$$

After dropping the total time derivative, we can finally get the Lagrangian

ANSWER TO PROBLEM 3 (B).

$$L = \frac{1}{2}ml^2 \dot{\phi}^2 + mla\gamma^2 \cos \gamma t \sin \phi + mgl \cos \phi$$

(c)

oscillates vertically according to the law $y = a \cos \gamma t$.

SOLUTION: The Cartesian position of the particle m is

$$\begin{aligned} x &= f(\phi_1) = l \sin \phi \\ y &= g(\phi_1) = a \cos \gamma t + l \cos \phi. \end{aligned}$$

The time derivative of the position is then

$$\begin{aligned} \dot{x} &= l\dot{\phi} \cos \phi \\ \dot{y} &= -a\gamma \sin \gamma t - l\dot{\phi} \sin \phi. \end{aligned}$$

The potential energy of the system is

$$U = -mgy = -mg(a \cos \gamma t + l \cos \phi),$$

The term $mga \cos \gamma t$ only depend on time and can therefore be ignored. The kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2}m \left(l^2 \dot{\phi}^2 + a^2 \gamma^2 \sin^2 \gamma t + 2a\gamma l \dot{\phi} \sin \gamma t \sin \phi \right). \end{aligned}$$

We first see that the term $\frac{1}{2}ma^2\gamma^2 \sin^2 \gamma t$ only depend on time. The last term of the kinetic energy can

also be simplified. Indeed,

$$\begin{aligned} m\gamma l \dot{\phi} \sin \gamma t \sin \phi &= mla\gamma(-\gamma \cos \gamma t \cos \phi + \dot{\phi} \sin \gamma t \sin \phi) + mla\gamma^2 \cos \gamma t \cos \phi \\ &= \frac{d}{dt}(mla\gamma \cos \gamma t \sin \phi) + mla\gamma^2 \cos \gamma t \cos \phi. \end{aligned}$$

After dropping the total time derivative, we can finally get the Lagrangian

ANSWER TO PROBLEM 3 (c).

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + mla\gamma^2 \cos \gamma t \cos \phi + mgl \cos \phi$$