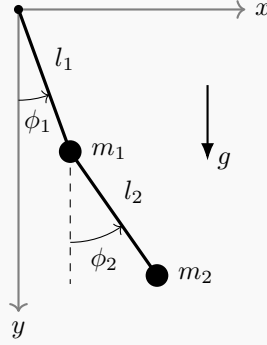


§5 THE LAGRANGIAN FOR A SYSTEM OF PARTICLES

PROBLEM 1.

Find the Lagrangian of a coplanar double pendulum when placed in a uniform gravitational field (acceleration g).



SOLUTION: The generalized co-ordinates of the system are the two angles ϕ_1 and ϕ_2 . We need to express the Cartesian co-ordinates in terms of those two angles. First, the Cartesian position of the particle m_1 is

$$\begin{aligned} x_1 &= f_1(\phi_1) = l_1 \sin \phi_1 \\ y_1 &= g_1(\phi_1) = l_1 \cos \phi_1. \end{aligned}$$

By taking the time derivative of those, we obtain

$$\begin{aligned} \dot{x}_1 &= \frac{\partial f_1}{\partial \phi_1} \dot{\phi}_1 = l_1 \cos \phi_1 \dot{\phi}_1 \\ \dot{y}_1 &= \frac{\partial g_1}{\partial \phi_1} \dot{\phi}_1 = -l_1 \sin \phi_1 \dot{\phi}_1. \end{aligned}$$

Substituting these expressions in the Lagrangian, we obtain the desired Lagrangian form, that is

$$\begin{aligned} L_1 &= T_1 - U_1 \\ &= \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + m_1 g y_1 \\ &= \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2 + m_1 g l_1 \cos \phi_1. \end{aligned} \tag{1.1}$$

Second, the Cartesian position of the particle m_2 is

$$\begin{aligned} x_2 &= f_2(\phi_1, \phi_2) = l_1 \sin \phi_1 + l_2 \sin \phi_2 \\ y_2 &= g_2(\phi_1, \phi_2) = l_1 \cos \phi_1 + l_2 \cos \phi_2. \end{aligned}$$

By taking the time derivative of those, we obtain

$$\begin{aligned} \dot{x}_2 &= \sum_k \frac{\partial f_2}{\partial \phi_k} \dot{\phi}_k = l_1 \dot{\phi}_1 \cos \phi_1 + l_2 \dot{\phi}_2 \cos \phi_2 \\ \dot{y}_2 &= \sum_k \frac{\partial g_2}{\partial \phi_k} \dot{\phi}_k = -l_1 \dot{\phi}_1 \sin \phi_1 - l_2 \dot{\phi}_2 \sin \phi_2. \end{aligned}$$

Substituting these expressions in the Lagrangian, we obtain the desired Lagrangian form, that is

$$\begin{aligned} L_2 &= T_2 - U_2 \\ &= \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) + m_2 g y_2 \\ &= \frac{1}{2} m_2 (l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)) + m_2 g (l_1 \cos \phi_1 + l_2 \cos \phi_2), \end{aligned} \tag{1.2}$$

where the angle difference identity as been used. Finally, the Lagrangian of the complete system is simply

the sum of Eq. (1.1) and Eq. (1.2), thus

ANSWER TO PROBLEM 1.

$$L = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2l_2^2\dot{\phi}_2^2 + m_2l_1l_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2) \\ + (m_1 + m_2)gl_1\cos\phi_1 + m_2gl_2\cos\phi_2$$