

§41 THE ROUTHIAN

PROBLEM 1.

Find the Routhian for a symmetrical top in an external field $U(\phi, \theta)$, eliminating the cyclic coordinate ψ (where ψ, ϕ, θ are Eulerian angles).

SOLUTION: Let $I'_1 = I'_2$ be the transverse moments of inertia about the fixed point and I_3 the moment of inertia about the axis of symmetry. In Eulerian angles, the Lagrangian is

$$L = \frac{I'_1}{2} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{I_3}{2} (\dot{\psi} + \dot{\phi} \cos \theta)^2 - U(\phi, \theta).$$

Since L is independent of ψ , its conjugate momentum is conserved:

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3 (\dot{\psi} + \dot{\phi} \cos \theta).$$

Solving for the velocity to be eliminated gives

$$\dot{\psi} = \frac{p_\psi}{I_3} - \dot{\phi} \cos \theta.$$

The Routhian is the partial Legendre transform $R = p_\psi \dot{\psi} - L$. Substituting the expression for $\dot{\psi}$, we get

$$R = p_\psi \left(\frac{p_\psi}{I_3} - \dot{\phi} \cos \theta \right) - \frac{I'_1}{2} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) - \frac{p_\psi^2}{2I_3} + U(\phi, \theta).$$

Thus, ψ and $\dot{\psi}$ are eliminated, while p_ψ is retained as a constant parameter.

ANSWER TO PROBLEM 1.

$$R(\phi, \theta, \dot{\phi}, \dot{\theta}; p_\psi) = \frac{p_\psi^2}{2I_3} - p_\psi \dot{\phi} \cos \theta - \frac{I'_1}{2} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + U(\phi, \theta)$$

For a given value of p_ψ , the term $p_\psi^2/2I_3$ is a constant, so it may be omitted when deriving the reduced equations of motion for ϕ and θ .