

PROBLEM 3.

Find the Hamiltonian for a system comprising one particle of mass M and n particles each of mass m , excluding the motion of the centre of mass (see §13 Prob. 1).

SOLUTION: We use the relative co-ordinates of §13 Prob. 1, namely $\mathbf{r}_a = \mathbf{R}_a - \mathbf{R}$ and $\mathbf{v}_a = \dot{\mathbf{r}}_a$, with $\mu \equiv M + nm$ the total mass. The reduced Lagrangian, including the potential energy, is

$$L = \frac{1}{2}m \sum_a \mathbf{v}_a^2 - \frac{1}{2} \frac{m^2}{\mu} \left(\sum_a \mathbf{v}_a \right)^2 - U.$$

The canonical momenta are

$$\mathbf{p}_a = \frac{\partial L}{\partial \mathbf{v}_a} = m\mathbf{v}_a - \frac{m^2}{\mu} \sum_b \mathbf{v}_b.$$

Summing over a , we get

$$\sum_a \mathbf{p}_a = \left(m - \frac{nm^2}{\mu} \right) \sum_a \mathbf{v}_a = \frac{mM}{\mu} \sum_a \mathbf{v}_a.$$

Substituting this relation back in the expression for $\mathbf{p}_a$ yields

$$\mathbf{v}_a = \frac{\mathbf{p}_a}{m} + \frac{1}{M} \sum_b \mathbf{p}_b.$$

Since the reduced kinetic energy is quadratic in the velocities, we have $\sum_a \mathbf{p}_a \cdot \mathbf{v}_a = 2T$ and

$$\begin{aligned} H = T + U &= \frac{1}{2} \sum_a \mathbf{p}_a \cdot \mathbf{v}_a + U \\ &= \frac{1}{2m} \sum_a \mathbf{p}_a^2 + \frac{1}{2M} \left(\sum_a \mathbf{p}_a \right)^2 + U. \end{aligned}$$

The momenta $\mathbf{p}_a$ are conjugate to the relative co-ordinates $\mathbf{r}_a$, and the kinetic energy of the centre of mass has already been removed.

ANSWER TO PROBLEM 3.

$$H = \frac{1}{2m} \sum_a \mathbf{p}_a^2 + \frac{1}{2M} \left(\sum_a \mathbf{p}_a \right)^2 + U(\mathbf{r}_1, \dots, \mathbf{r}_n)$$