

§40 HAMILTON'S EQUATIONS

PROBLEM 1.

Find the Hamiltonian for a single particle in Cartesian, cylindrical and spherical co-ordinates.

SOLUTION: For a particle with a velocity-independent potential energy U , the generalized momenta and the Hamiltonian are

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad H = \sum_i p_i \dot{q}_i - L.$$

The kinetic energy is quadratic in the velocities, so the Legendre transform gives $H = T + U$, with the velocities expressed in terms of the canonical momenta.

In Cartesian co-ordinates, we have

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z), \quad p_x = m\dot{x}, \quad p_y = m\dot{y}, \quad p_z = m\dot{z}.$$

In cylindrical co-ordinates, the Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) - U(r, \phi, z),$$

so that

$$p_r = m\dot{r}, \quad p_\phi = mr^2\dot{\phi}, \quad p_z = m\dot{z}.$$

In spherical co-ordinates, with θ the polar angle and ϕ the azimuth, the Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2 \theta) - U(r, \theta, \phi),$$

and hence

$$p_r = m\dot{r}, \quad p_\theta = mr^2\dot{\theta}, \quad p_\phi = mr^2\dot{\phi} \sin^2 \theta.$$

Substituting these momenta in $H = T + U$ gives the three Hamiltonians.

ANSWER TO PROBLEM 1.

$$\text{Cartesian: } H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + U(x, y, z)$$

$$\text{Cylindrical: } H = \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2} + p_z^2\right) + U(r, \phi, z)$$

$$\text{Spherical: } H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta}\right) + U(r, \theta, \phi)$$