

PROBLEM 6.

Determine the form of a curve such that the frequency of oscillations of a particle on it under the force of gravity is independent of the amplitude.

SOLUTION: Consider a particle constrained to slide without friction along a plane curve in a uniform gravitational field. We set the lowest point at $x = y = 0$, with y measured vertically upwards, and let s be the signed arc length along the curve, measured from that point.

The arc length is a useful generalized co-ordinate because the speed is exactly $|\dot{s}|$. Thus,

$$T = \frac{1}{2}m\dot{s}^2, \quad U = mgy(s),$$

and the equation of motion is

$$\ddot{s} + g \frac{dy}{ds} = 0.$$

To construct the required curve, we require the motion in s to be exactly harmonic with a prescribed constant angular frequency ω :

$$\ddot{s} + \omega^2 s = 0.$$

This is sufficient to make the frequency independent of the amplitude. Comparing the two equations and integrating, with $y(0) = 0$, gives

$$g \frac{dy}{ds} = \omega^2 s, \quad y = \frac{\omega^2 s^2}{2g}.$$

No small-amplitude expansion has been used.

The required relation between height and arc length now determines the geometry. On the right-hand branch, $s > 0$, we have

$$s = \frac{\sqrt{2gy}}{\omega}, \quad \left(\frac{ds}{dy} \right)^2 = \frac{g}{2\omega^2 y}.$$

Since $ds^2 = dx^2 + dy^2$,

$$\left(\frac{dx}{dy} \right)^2 = \left(\frac{ds}{dy} \right)^2 - 1 = \frac{g}{2\omega^2 y} - 1.$$

We introduce the length $a = g/(4\omega^2)$. Then,

$$\frac{dx}{dy} = \sqrt{\frac{2a - y}{y}}.$$

We use the substitution

$$y = a(1 - \cos \theta), \quad 0 \leq \theta < \pi,$$

for which

$$dy = a \sin \theta d\theta, \quad \sqrt{\frac{2a - y}{y}} = \cot \frac{\theta}{2}.$$

Therefore,

$$dx = a \sin \theta \cot \frac{\theta}{2} d\theta = a(1 + \cos \theta) d\theta.$$

Integrating from the lowest point yields

$$x = a(\theta + \sin \theta), \quad y = a(1 - \cos \theta).$$

Reflection gives the left-hand branch, represented by the same formulae with negative θ . This is a cycloidal trough, with its lowest point at $\theta = 0$ and adjacent cusps at $\theta = \pm\pi$.

We now verify the exact independence of the amplitude explicitly. On the smooth arc $-\pi < \theta < \pi$,

$$\begin{aligned}\left(\frac{ds}{d\theta}\right)^2 &= \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 \\ &= a^2(1 + \cos \theta)^2 + a^2 \sin^2 \theta \\ &= 4a^2 \cos^2 \frac{\theta}{2}.\end{aligned}$$

Choosing s to increase with θ , we get

$$s = 4a \sin \frac{\theta}{2}.$$

It follows that

$$y = 2a \sin^2 \frac{\theta}{2} = \frac{s^2}{8a}.$$

Thus, the exact Lagrangian becomes

$$L = \frac{1}{2}m\dot{s}^2 - \frac{mg}{8a}s^2,$$

and the exact equation of motion is

$$\ddot{s} + \frac{g}{4a}s = 0.$$

Hence, every oscillation with amplitude $|s| < 4a$, *i.e.* whose turning points lie below the height $2a$ of the cusps, has the same angular frequency $\omega = \sqrt{g/4a}$ and period $\tau = 2\pi/\omega$. This applies to finite amplitudes, not merely to infinitesimal oscillations.

ANSWER TO PROBLEM 6.

The curve is a cycloidal trough, where $a > 0$ is the radius of its generating circle, with angular frequency ω and period τ :

$$\begin{aligned}x &= a(\theta + \sin \theta), \\ y &= a(1 - \cos \theta), \quad -\pi < \theta < \pi, \\ \omega &= \sqrt{\frac{g}{4a}}, \quad \tau = 4\pi\sqrt{\frac{a}{g}}.\end{aligned}$$