

PROBLEM 5.

Find the frequency of oscillations of the pendulum shown in the figure in §5 Prob. 2, whose point of support carries a mass m_1 and is free to move horizontally (its motion is integrated in §14 Prob. 3).

SOLUTION: Let x denote the horizontal position of the support and ϕ the angle of the pendulum from the downward vertical. The support has mass m_1 , the bob has mass m_2 , and the length of the pendulum is l .

From §5 Prob. 2, the Lagrangian is

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2l^2\dot{\phi}^2 + m_2l\dot{x}\dot{\phi}\cos\phi + m_2gl\cos\phi.$$

Because x is cyclic, the total horizontal momentum is conserved:

$$p_x = \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x} + m_2l\dot{\phi}\cos\phi.$$

A constant horizontal translation of the whole system does not affect its oscillation frequency. We may therefore work in the inertial frame in which $p_x = 0$, as in §14 Prob. 3. In this frame,

$$\dot{x} = -\frac{m_2l}{m_1 + m_2}\dot{\phi}\cos\phi.$$

Substituting this expression into the kinetic energy, we get

$$\begin{aligned} T &= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2l^2\dot{\phi}^2 + m_2l\dot{x}\dot{\phi}\cos\phi \\ &= \frac{m_2^2l^2}{2(m_1 + m_2)}\dot{\phi}^2\cos^2\phi + \frac{1}{2}m_2l^2\dot{\phi}^2 - \frac{m_2^2l^2}{m_1 + m_2}\dot{\phi}^2\cos^2\phi \\ &= \frac{1}{2}m_2l^2\left(1 - \frac{m_2}{m_1 + m_2}\cos^2\phi\right)\dot{\phi}^2. \end{aligned}$$

For small oscillations, we retain terms of second order in the displacement and velocity. Since $\dot{\phi}^2$ is already of second order, replacing $\cos^2\phi$ by 1 changes T only at fourth order. Hence,

$$T \simeq \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}l^2\dot{\phi}^2.$$

Taking the gravitational potential energy to vanish at $\phi = 0$, we have

$$U = m_2gl(1 - \cos\phi) \simeq \frac{1}{2}m_2gl\phi^2.$$

Eliminating a cyclic velocity directly in the Lagrangian is not legitimate in general. The correct Lagrangian for the remaining co-ordinate is $L - p_x\dot{x}$ (minus the Routhian), with $\dot{x}$ expressed in terms of p_x and ϕ . Here, $p_x = 0$, so the extra term vanishes and the Lagrangian for ϕ is simply $T - U$ with $\dot{x}$ eliminated. Equivalently, the energy $E = T + U$ then takes the one-dimensional form $\frac{1}{2}M(\phi)\dot{\phi}^2 + U(\phi)$, which determines the frequency.

The quadratic Lagrangian for the oscillation is therefore

$$L^{(2)} = \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}l^2\dot{\phi}^2 - \frac{1}{2}m_2gl\phi^2.$$

Its Euler–Lagrange equation gives

$$\frac{m_1m_2}{m_1 + m_2}l^2\ddot{\phi} + m_2gl\phi = 0,$$

or

$$\ddot{\phi} + \frac{g(m_1 + m_2)}{m_1l}\phi = 0.$$

The freedom of the support to recoil reduces the effective inertia of the oscillatory co-ordinate. Consequently, the frequency exceeds that of a pendulum with a fixed support. In the limit $m_1 \rightarrow \infty$, the

fixed-support result $\sqrt{g/l}$ is recovered.

ANSWER TO PROBLEM 5.

$$\omega = \sqrt{\frac{g(m_1 + m_2)}{m_1 l}}$$