

PROBLEM 4.

The same as §21 Prob. 3, but for a particle of mass m moving on a circle of radius r (Fig. 23).

SOLUTION: Let O be the centre of the circle of radius r . The fixed end A lies outside the circle, with $OA = l + r$, so that the spring has length l when the particle is at the point nearest A . (If A were inside the circle, with $OA = r - l$, the same steps would give $\omega^2 = F(r - l)/(mrl)$.) Let ϕ be the angle between OA and the radius to the particle.

By the cosine rule, the length of the spring is

$$\begin{aligned} d(\phi) &= \sqrt{r^2 + (l + r)^2 - 2r(l + r)\cos\phi} \\ &= \sqrt{l^2 + 2r(l + r)(1 - \cos\phi)}. \end{aligned}$$

For small oscillations, $1 - \cos\phi \simeq \phi^2/2$. Factoring out l and using $\sqrt{1 + \epsilon} \simeq 1 + \epsilon/2$, we get

$$d = l\sqrt{1 + \frac{2r(l + r)}{l^2}(1 - \cos\phi)} \simeq l + \frac{r(l + r)}{2l}\phi^2,$$

and therefore

$$\delta l = d(\phi) - l \simeq \frac{r(l + r)}{2l}\phi^2.$$

The amplitude must be small enough for both expansions to hold: $\phi \ll 1$ for the cosine, and $r(l + r)\phi^2/l^2 \ll 1$ for the square root. As in §21 Prob. 3, the leading change in the energy of the spring is $F\delta l$. Hence,

$$U(\phi) \simeq \frac{Fr(l + r)}{2l}\phi^2.$$

The speed of the particle is $r\dot{\phi}$, giving

$$T = \frac{1}{2}mr^2\dot{\phi}^2.$$

The linearized equation of motion is

$$mr^2\ddot{\phi} + \frac{Fr(l + r)}{l}\phi = 0,$$

or

$$\ddot{\phi} + \frac{F(l + r)}{mrl}\phi = 0.$$

As a consistency check, at fixed l the circle becomes a straight line at distance l from A as $r \rightarrow \infty$, and

$$\omega^2 = \frac{F}{ml} \left(1 + \frac{l}{r}\right) \rightarrow \frac{F}{ml} \quad \text{as } r \rightarrow \infty,$$

which recovers the result of §21 Prob. 3.

ANSWER TO PROBLEM 4.

$$\omega = \sqrt{\frac{F(l + r)}{mrl}}$$