

PROBLEM 3.

Find the frequency of oscillations of a particle of mass m which is free to move along a line and is attached to a spring whose other end is fixed at a point A (Fig. 22) at a distance l from the line. A force F is required to extend the spring to length l .

SOLUTION: Let x be the displacement along the line from the point nearest the fixed end A . The length of the spring is

$$d(x) = \sqrt{l^2 + x^2}.$$

Here, l is the length of the spring in the constrained equilibrium configuration $x = 0$, where the spring is already stretched and carries a tension $F > 0$; it is not the natural length of the spring. For $|x| \ll l$, the additional extension from this equilibrium length is

$$\delta l = d(x) - l = l \left(\sqrt{1 + \frac{x^2}{l^2}} - 1 \right) \simeq \frac{x^2}{2l}.$$

Let $V(d)$ denote the potential energy of the spring. The force required to hold it at length l is $F = V'(l) > 0$. Expanding about that length and dropping the constant $V(l)$, we get

$$U(x) = V(l + \delta l) - V(l) = F \delta l + \frac{1}{2} V''(l) (\delta l)^2 + \dots$$

Since δl is already quadratic in x , its square contributes only at fourth order. Thus, the harmonic approximation gives

$$U(x) \simeq \frac{F}{2l} x^2.$$

The restoring potential therefore comes from the first-order term $F \delta l$, *i.e.* from the tension already present at equilibrium, and not from the stiffness $V''(l)$ of the spring.

The kinetic energy is

$$T = \frac{1}{2} m \dot{x}^2,$$

and the equation of motion becomes

$$m \ddot{x} + \frac{F}{l} x = 0.$$

As an independent check, the tension $V'(d)$ pulls the particle towards A along the spring, and its component along the line is

$$F_x = -V'(d) \frac{x}{d} \simeq -\frac{F}{l} x$$

to first order in x , which reproduces this equation of motion. Comparison with $\ddot{x} + \omega^2 x = 0$ gives the angular frequency.

ANSWER TO PROBLEM 3.

$$\omega = \sqrt{\frac{F}{ml}}$$