

§21 FREE OSCILLATIONS IN ONE DIMENSION

PROBLEM 1.

Express the amplitude and initial phase of the oscillations in terms of the initial co-ordinate x_0 and velocity v_0 .

SOLUTION: The general solution of $\ddot{x} + \omega^2 x = 0$, with angular frequency $\omega > 0$, that satisfies $x(0) = x_0$ and $\dot{x}(0) = v_0$ is

$$x(t) = x_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t.$$

We write the same harmonic motion as

$$x(t) = a \cos(\omega t + \alpha) = a \cos \alpha \cos \omega t - a \sin \alpha \sin \omega t, \quad a \geq 0.$$

Comparing the coefficients of $\cos \omega t$ and $\sin \omega t$, we get

$$a \cos \alpha = x_0, \quad a \sin \alpha = -\frac{v_0}{\omega}.$$

These are the relations $x_0 = a \cos \alpha$ and $v_0 = -a\omega \sin \alpha$ obtained by evaluating x and $\dot{x}$ directly at $t = 0$. Squaring them and adding gives

$$x_0^2 + \frac{v_0^2}{\omega^2} = a^2 (\cos^2 \alpha + \sin^2 \alpha) = a^2.$$

For a nonzero amplitude, the phase is fixed modulo 2π by

$$\cos \alpha = \frac{x_0}{a}, \quad \sin \alpha = -\frac{v_0}{a\omega}.$$

Thus, $\tan \alpha = -v_0/(\omega x_0)$ when $x_0 \neq 0$, with the quadrant determined by the signs of the sine and cosine. The two-argument arctangent $\text{atan2}(Y, X)$ returns the angle whose cosine and sine have the signs of X and Y respectively, and so selects the correct quadrant. If $x_0 = v_0 = 0$, the particle remains at equilibrium and the phase is arbitrary.

ANSWER TO PROBLEM 1.

$$a = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}, \quad \alpha = \text{atan2}\left(-\frac{v_0}{\omega}, x_0\right) \pmod{2\pi}$$