

PROBLEM 2.

Determine the effective cross-section for small-angle scattering in a field $U = \alpha/r^n$ ($n > 0$).

SOLUTION: We take $\alpha > 0$ (a repulsive field). Let m_2 be the mass of the scattering particle, initially at rest, and

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

the reduced mass. As shown in §20 Prob. 1, when the scattering angle χ in the centre-of-mass system is small ($\chi \ll 1$), the deflection of the incident particle of mass m_1 is $\theta_1 \simeq (m/m_1)\chi$, and it is given by formula (20.3) of the book,

$$\theta_1 \simeq -\frac{2\rho}{m_1 v_\infty^2} \int_\rho^\infty \frac{dU}{dr} \frac{dr}{\sqrt{r^2 - \rho^2}}.$$

Since

$$\frac{dU}{dr} = -\frac{n\alpha}{r^{n+1}},$$

the deflection is

$$\theta_1 \simeq \frac{2n\alpha\rho}{m_1 v_\infty^2} \int_\rho^\infty \frac{dr}{r^{n+1} \sqrt{r^2 - \rho^2}}.$$

To evaluate the integral, we substitute

$$u = \frac{\rho^2}{r^2}, \quad r = \rho u^{-1/2}, \quad dr = -\frac{\rho}{2} u^{-3/2} du.$$

When r goes from ρ to ∞ , u goes from 1 to 0, and the minus sign of dr is absorbed by reversing the limits. We also have

$$r^{n+1} = \rho^{n+1} u^{-(n+1)/2}, \quad \sqrt{r^2 - \rho^2} = \rho \sqrt{\frac{1}{u} - 1} = \rho u^{-1/2} (1 - u)^{1/2}.$$

Collecting the powers of u , $-\frac{3}{2} + \frac{n+1}{2} + \frac{1}{2} = \frac{n-1}{2}$, so

$$\int_\rho^\infty \frac{dr}{r^{n+1} \sqrt{r^2 - \rho^2}} = \frac{1}{2\rho^{n+1}} \int_0^1 u^{(n-1)/2} (1-u)^{-1/2} du.$$

As in §11 Prob. 2, this is the Beta function $B(p, q) = \int_0^1 u^{p-1} (1-u)^{q-1} du = \Gamma(p)\Gamma(q)/\Gamma(p+q)$ with $p = (n+1)/2$ and $q = 1/2$. Since $\Gamma(1/2) = \sqrt{\pi}$,

$$\int_\rho^\infty \frac{dr}{r^{n+1} \sqrt{r^2 - \rho^2}} = \frac{1}{2\rho^{n+1}} B\left(\frac{n+1}{2}, \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2\rho^{n+1}} \frac{\Gamma((n+1)/2)}{\Gamma((n+2)/2)}.$$

Using $\Gamma((n+2)/2) = \frac{n}{2}\Gamma(n/2)$, the factor n cancels and

$$\theta_1 \simeq \frac{2\sqrt{\pi}\alpha}{m_1 v_\infty^2 \rho^n} \frac{\Gamma((n+1)/2)}{\Gamma(n/2)}.$$

Let us define

$$C_n = \frac{2\sqrt{\pi}\alpha}{m_1 v_\infty^2} \frac{\Gamma((n+1)/2)}{\Gamma(n/2)},$$

so that $\theta_1 \simeq C_n/\rho^n$. The deflection decreases monotonically with ρ , and the relation can be inverted,

$$\rho \simeq \left(\frac{C_n}{\theta_1}\right)^{1/n}, \quad \left|\frac{d\rho}{d\theta_1}\right| = \frac{1}{n} C_n^{1/n} \theta_1^{-1/n-1} = \frac{\rho}{n\theta_1}.$$

The particles deflected into the range $d\theta_1$ pass through an annulus of area $2\pi\rho|d\rho|$ in the incident beam. For small angles, the element of solid angle is

$$d\Omega_1 = 2\pi \sin \theta_1 d\theta_1 \simeq 2\pi \theta_1 d\theta_1,$$

so

$$\frac{d\sigma}{d\Omega_1} \simeq \frac{\rho}{\theta_1} \left|\frac{d\rho}{d\theta_1}\right| = \frac{\rho^2}{n\theta_1^2} = \frac{1}{n} \frac{C_n^{2/n}}{\theta_1^{2+2/n}}.$$

This result requires weak scattering in the centre-of-mass system, $\chi \simeq (m_1/m)\theta_1 \ll 1$. Multiplying

$\theta_1 \simeq C_n/\rho^n$ by m_1/m , the corresponding centre-of-mass deflection is

$$\chi \simeq \frac{2\sqrt{\pi} \Gamma((n+1)/2)}{\Gamma(n/2)} \frac{\alpha/\rho^n}{mv_\infty^2},$$

whose numerical coefficient is of order unity for n of order unity. The condition $\chi \ll 1$ is thus the weak-field condition

$$\frac{\alpha}{\rho^n} \ll \frac{1}{2}mv_\infty^2,$$

i.e. the potential energy at the distance of closest approach $r \simeq \rho$ must be small compared with the energy of the relative motion. Since $m \leq m_1$, it implies $\theta_1 \ll 1$, which justifies $do_1 \simeq 2\pi\theta_1 d\theta_1$. However, the converse does not hold for arbitrary masses. For instance, if $m_1 > m_2$, the laboratory angle never exceeds $\arcsin(m_2/m_1)$, so it can be small even when χ is not. The two conditions coincide only when $m_2 \gg m_1$, where $m \simeq m_1$ and $\theta_1 \simeq \chi$. The result must not be extended to arbitrary angles. Note also that $d\sigma$ diverges as $\theta_1 \rightarrow 0$, so the total cross-section is infinite unless the angles are cut off.

As a check, for $n = 1$ we have $\Gamma(1)/\Gamma(1/2) = 1/\sqrt{\pi}$, so $C_1 = 2\alpha/(m_1v_\infty^2)$ and

$$\frac{d\sigma}{do_1} \simeq \frac{4\alpha^2}{m_1^2v_\infty^4\theta_1^4},$$

which is the small-angle limit of the Rutherford formula. For $n = 2$, $\Gamma(3/2)/\Gamma(1) = \sqrt{\pi}/2$, so $C_2 = \pi\alpha/(m_1v_\infty^2)$ and

$$\frac{d\sigma}{do_1} \simeq \frac{\pi\alpha}{2m_1v_\infty^2\theta_1^3},$$

in agreement with the small-angle limit found in §19 Prob. 1 for a fixed centre ($m_1 = m$, $\theta_1 = \chi$). We thus get

ANSWER TO PROBLEM 2.

$$\begin{aligned} \theta_1 &\simeq \frac{2\sqrt{\pi} \Gamma((n+1)/2)}{\Gamma(n/2)} \frac{\alpha}{m_1v_\infty^2\rho^n}, \\ d\sigma &\simeq \frac{1}{n} \left[\frac{2\sqrt{\pi} \Gamma((n+1)/2)}{\Gamma(n/2)} \frac{\alpha}{m_1v_\infty^2} \right]^{2/n} \frac{do_1}{\theta_1^{2+2/n}}, \\ \text{valid for } \frac{\alpha}{\rho^n} &\ll \frac{1}{2}mv_\infty^2, \quad m = \frac{m_1m_2}{m_1+m_2} \end{aligned}$$