

§20 SMALL-ANGLE SCATTERING

PROBLEM 1.

Derive formula (20.3) from (18.4).

SOLUTION: Consider the relative motion, with reduced mass m and relative velocity at infinity v_∞ . Formula (18.4) of the book gives

$$\phi_0 = \int_{r_{\min}}^{\infty} \frac{\rho \, dr}{r^2 \sqrt{1 - \frac{\rho^2}{r^2} - \frac{2U(r)}{mv_\infty^2}}}, \quad \chi = \pi - 2\phi_0.$$

For a weak field, it is tempting to expand the integrand in powers of U . However, the integrand behaves as $1/\sqrt{r - r_{\min}}$ near the turning point. Expanding it gives terms like $U/(1 - \rho^2/r^2)^{3/2}$, which are not integrable at $r = \rho$. These divergences are spurious: they only come from expanding near the point where the radicand vanishes.

To avoid this, we write ϕ_0 as the derivative of an integral whose integrand vanishes at the turning point. Since this integral does not converge at infinity, we first introduce a finite upper limit R ,

$$J_R(\rho) = \int_{r_{\min}}^R \sqrt{1 - \frac{\rho^2}{r^2} - \frac{2U(r)}{mv_\infty^2}} \, dr.$$

We differentiate with respect to ρ at fixed R . The lower limit $r_{\min}$ depends on ρ , but its contribution is the integrand at $r = r_{\min}$ times $dr_{\min}/d\rho$, which vanishes since the square root is zero there. Hence

$$\frac{\partial J_R}{\partial \rho} = - \int_{r_{\min}}^R \frac{\rho \, dr}{r^2 \sqrt{1 - \frac{\rho^2}{r^2} - \frac{2U(r)}{mv_\infty^2}}} \Rightarrow \phi_{0,R} = - \frac{\partial J_R}{\partial \rho},$$

where $\phi_{0,R}$ is the integral (18.4) cut off at $r = R$. We keep R finite until after the differentiation, since J_R itself grows like R and has no limit as $R \rightarrow \infty$.

We now expand J_R to first order in U . Pointwise,

$$\sqrt{1 - \frac{\rho^2}{r^2} - \frac{2U}{mv_\infty^2}} \simeq \sqrt{1 - \frac{\rho^2}{r^2}} - \frac{U}{mv_\infty^2 \sqrt{1 - \rho^2/r^2}} = \sqrt{1 - \frac{\rho^2}{r^2}} - \frac{1}{mv_\infty^2} \frac{U(r) r}{\sqrt{r^2 - \rho^2}}.$$

This expansion is not uniform near the turning point, but it can be integrated. To see why, regard J_R as a function of the strength of the field. Its first-order variation contains a term from the shift of the lower limit, from ρ (the turning point when $U = 0$) to $r_{\min}$. This term is proportional to the unperturbed integrand at its lower endpoint, which vanishes. The other term is the integral of the first-order variation of the integrand, which has only an integrable singularity $1/\sqrt{r - \rho}$. To first order, we can therefore take the lower limit at ρ ,

$$J_R \simeq \int_{\rho}^R \sqrt{1 - \frac{\rho^2}{r^2}} \, dr - \frac{1}{mv_\infty^2} \int_{\rho}^R \frac{U(r) r \, dr}{\sqrt{r^2 - \rho^2}} = \int_{\rho}^R \sqrt{1 - \frac{\rho^2}{r^2}} \, dr - \frac{K_R(\rho)}{mv_\infty^2},$$

where

$$K_R(\rho) = \int_{\rho}^R \frac{U(r) r \, dr}{\sqrt{r^2 - \rho^2}}.$$

The free term can be integrated directly,

$$\int_{\rho}^R \sqrt{1 - \frac{\rho^2}{r^2}} \, dr = \int_{\rho}^R \frac{\sqrt{r^2 - \rho^2}}{r} \, dr = \left[\sqrt{r^2 - \rho^2} - \rho \arccos \frac{\rho}{r} \right]_{\rho}^R = \sqrt{R^2 - \rho^2} - \rho \arccos \frac{\rho}{R},$$

as can be checked by differentiating the bracket with respect to r ,

$$\frac{r}{\sqrt{r^2 - \rho^2}} - \frac{\rho^2}{r \sqrt{r^2 - \rho^2}} = \frac{\sqrt{r^2 - \rho^2}}{r}.$$

Its derivative with respect to ρ is

$$\frac{\partial}{\partial \rho} \left(\sqrt{R^2 - \rho^2} - \rho \arccos \frac{\rho}{R} \right) = -\frac{\rho}{\sqrt{R^2 - \rho^2}} - \arccos \frac{\rho}{R} + \frac{\rho}{\sqrt{R^2 - \rho^2}} = -\arccos \frac{\rho}{R}.$$

Consequently,

$$\phi_{0,R} \simeq \arccos \frac{\rho}{R} + \frac{1}{mv_\infty^2} \frac{\partial K_R}{\partial \rho}.$$

We cannot differentiate K_R directly under the integral sign, because its integrand is singular at the lower limit $r = \rho$. We therefore integrate by parts first, using

$$\frac{r \, dr}{\sqrt{r^2 - \rho^2}} = d\sqrt{r^2 - \rho^2},$$

which gives

$$K_R = \left[U(r) \sqrt{r^2 - \rho^2} \right]_\rho^R - \int_\rho^R U'(r) \sqrt{r^2 - \rho^2} \, dr = U(R) \sqrt{R^2 - \rho^2} - \int_\rho^R U'(r) \sqrt{r^2 - \rho^2} \, dr.$$

The boundary term at $r = \rho$ vanishes since $\sqrt{r^2 - \rho^2} = 0$ there. The new integrand is finite at $r = \rho$, and it vanishes there. Hence, when we differentiate with respect to ρ , the variation of the lower limit again gives no contribution, and

$$\frac{\partial K_R}{\partial \rho} = -\frac{\rho U(R)}{\sqrt{R^2 - \rho^2}} + \rho \int_\rho^R \frac{U'(r) \, dr}{\sqrt{r^2 - \rho^2}}.$$

Now we take $R \rightarrow \infty$. Then $\arccos(\rho/R) \rightarrow \pi/2$, and the first term vanishes since $U(R) \rightarrow 0$ and $\rho/\sqrt{R^2 - \rho^2} \rightarrow 0$. Assuming the remaining integral converges (*i.e.* the force decreases fast enough), we get

$$\phi_0 \simeq \frac{\pi}{2} + \frac{\rho}{mv_\infty^2} \int_\rho^\infty \frac{U'(r) \, dr}{\sqrt{r^2 - \rho^2}},$$

and therefore

$$\chi = \pi - 2\phi_0 \simeq -\frac{2\rho}{mv_\infty^2} \int_\rho^\infty \frac{U'(r) \, dr}{\sqrt{r^2 - \rho^2}}.$$

For a repulsive field, $U' < 0$ and $\chi > 0$, as it should be. This result is valid only if the deflection is small, *i.e.* if the field is weak along the whole path ($|U| \ll \frac{1}{2}mv_\infty^2$ for $r \gtrsim \rho$). Under this condition, the first-order expansion is justified.

The angle χ is the scattering angle in the centre-of-mass system. Formula (20.3) gives instead θ_1 , the deflection of the incident particle in the laboratory system, where the particle m_2 is initially at rest. From formula (17.4) of the book, for small angles

$$\tan \theta_1 = \frac{m_2 \sin \chi}{m_1 + m_2 \cos \chi} \Rightarrow \theta_1 \simeq \frac{m_2}{m_1 + m_2} \chi = \frac{m}{m_1} \chi, \quad m = \frac{m_1 m_2}{m_1 + m_2}.$$

Multiplying χ by m/m_1 , the reduced mass cancels. We thus get

ANSWER TO PROBLEM 1.

$$\begin{aligned} \theta_1 &\simeq -\frac{2\rho}{m_1 v_\infty^2} \int_\rho^\infty \frac{dU}{dr} \frac{dr}{\sqrt{r^2 - \rho^2}}, \\ \chi &\simeq -\frac{2\rho}{mv_\infty^2} \int_\rho^\infty \frac{dU}{dr} \frac{dr}{\sqrt{r^2 - \rho^2}} \end{aligned}$$