

PROBLEM 2.

Find the effective cross-section for scattering by a spherical “potential well” of radius a and “depth” U_0 (*i.e.* a field with $U = 0$ for $r > a$ and $U = -U_0$ for $r < a$).

SOLUTION: Let m be the mass of the particle and v_∞ its velocity at infinity. Since U is constant outside the sphere, the particle moves with the constant velocity v_∞ until it reaches $r = a$. Inside the well, U is again constant, so the particle moves with a constant velocity v_{in} given by the conservation of energy,

$$\frac{1}{2}mv_{\text{in}}^2 - U_0 = \frac{1}{2}mv_\infty^2.$$

As in the book, we define the ratio

$$n = \frac{v_{\text{in}}}{v_\infty} = \sqrt{1 + \frac{2U_0}{mv_\infty^2}} > 1.$$

The trajectory is therefore made of straight segments, and its direction changes only at the boundary $r = a$, where the particle receives an impulse directed along the radius. This is analogous to the refraction of a light ray by a sphere with refractive index n .

The geometry of the path is shown in Fig. 21 of the book. Let α be the angle of incidence and β the angle of refraction, both measured from the normal to the sphere (*i.e.* the radius) at the point of entry. At $r = a$, the angular momentum is conserved (the force is radial), so the component of the velocity perpendicular to the radius is the same on both sides,

$$mv_\infty a \sin \alpha = mv_{\text{in}} a \sin \beta \Rightarrow \frac{\sin \alpha}{\sin \beta} = n.$$

Particles with $\rho > a$ do not touch the sphere and are not deflected. For $\rho < a$, the impact parameter is the distance from the centre to the incident line, so

$$\rho = a \sin \alpha.$$

At the entry, the direction of motion turns towards the centre by an angle $\alpha - \beta$. Inside the sphere, the particle moves along a chord, which makes the same angle β with the radii at both of its ends. By the symmetry of the path with respect to the radius bisecting this chord, the particle leaves the sphere at the angle α with the normal and is again deflected by $\alpha - \beta$ in the same sense. The total angle of deflection is thus

$$\chi = 2(\alpha - \beta).$$

To express ρ in terms of χ , let $q = \chi/2$, so that $\beta = \alpha - q$. The law of refraction becomes

$$\sin(\alpha - q) = \sin \alpha \cos q - \cos \alpha \sin q = \frac{\sin \alpha}{n}.$$

Dividing by $\sin \alpha$,

$$\cos q - \cot \alpha \sin q = \frac{1}{n} \Rightarrow \cot \alpha = \frac{n \cos q - 1}{n \sin q}.$$

Using $\sin^2 \alpha = 1/(1 + \cot^2 \alpha)$, we get

$$\sin^2 \alpha = \frac{n^2 \sin^2 q}{n^2 \sin^2 q + (n \cos q - 1)^2} = \frac{n^2 \sin^2 q}{n^2 + 1 - 2n \cos q},$$

since $n^2 \sin^2 q + n^2 \cos^2 q - 2n \cos q + 1 = n^2 + 1 - 2n \cos q$. Therefore

$$\rho^2 = a^2 \sin^2 \alpha = \frac{a^2 n^2 \sin^2(\chi/2)}{n^2 + 1 - 2n \cos(\chi/2)}.$$

Let $D = n^2 + 1 - 2n \cos q$, so that $dD/dq = 2n \sin q$. Since $dq/d\chi = 1/2$, the quotient rule gives

$$\frac{d(\rho^2)}{d\chi} = \frac{a^2 n^2}{2} \frac{2 \sin q \cos q D - \sin^2 q \cdot 2n \sin q}{D^2} = \frac{a^2 n^2 \sin q [\cos q D - n \sin^2 q]}{D^2}.$$

The bracket factorizes as

$$\begin{aligned}
\cos q D - n \sin^2 q &= n^2 \cos q + \cos q - 2n \cos^2 q - n + n \cos^2 q \\
&= n^2 \cos q - n \cos^2 q - n + \cos q \\
&= (n \cos q - 1)(n - \cos q).
\end{aligned}$$

We now check that χ increases with ρ . As ρ increases from 0 to a , α increases from 0 to $\pi/2$, and since $\sin \beta = \sin \alpha/n$,

$$\frac{d\chi}{d\alpha} = 2 \left(1 - \frac{d\beta}{d\alpha} \right) = 2 \left(1 - \frac{\cos \alpha}{n \cos \beta} \right) = 2 \left(1 - \frac{\cos \alpha}{\sqrt{n^2 - \sin^2 \alpha}} \right) > 0,$$

because $n^2 - \sin^2 \alpha > 1 - \sin^2 \alpha = \cos^2 \alpha$. The relation between ρ and χ is thus one-to-one, and $d(\rho^2)/d\chi > 0$. The particles deflected into the range $d\chi$ pass through an annulus of area

$$d\sigma = \pi \frac{d(\rho^2)}{d\chi} d\chi = \frac{1}{2 \sin \chi} \frac{d(\rho^2)}{d\chi} d\chi,$$

where we used $d\phi = 2\pi \sin \chi d\chi$. With $\sin \chi = 2 \sin q \cos q$, the factor $\sin q$ cancels, and

$$d\sigma = \frac{a^2 n^2 (n \cos q - 1)(n - \cos q)}{4 \cos q D^2} d\phi.$$

Let us find the range of χ . For $\rho = 0$, the particle crosses the sphere along a diameter, so $\alpha = \beta = 0$ and $\chi = 0$. As $\rho \rightarrow a$ from below, the particle grazes the sphere, with

$$\alpha \rightarrow \frac{\pi}{2}, \quad \beta \rightarrow \arcsin \frac{1}{n}.$$

Hence

$$\chi_{max} = 2 \left(\frac{\pi}{2} - \arcsin \frac{1}{n} \right) = 2 \arccos \frac{1}{n}, \quad \cos \frac{\chi_{max}}{2} = \frac{1}{n}.$$

The formula for $d\sigma$ applies for $0 < \chi < \chi_{max}$. Accordingly, the factor $n \cos(\chi/2) - 1$ is positive in this range and vanishes at $\chi = \chi_{max}$. Particles with $\rho > a$ are not deflected and do not contribute to the scattering.

Finally, the total cross-section for scattering is obtained directly from the area of the annuli,

$$\sigma = \int_0^{\chi_{max}} d\sigma = \pi [\rho^2(\chi_{max}) - \rho^2(0)].$$

At $\chi = \chi_{max}$, $\cos q = 1/n$, so $\sin^2 q = (n^2 - 1)/n^2$ and $D = n^2 - 1$, which gives $\rho^2(\chi_{max}) = a^2$. Since $\rho^2(0) = 0$, $\sigma = \pi a^2$. This is simply the cross-sectional area of the sphere: every particle which hits the sphere is deflected. We thus get

ANSWER TO PROBLEM 2.

$$\begin{aligned}
d\sigma &= \frac{a^2 n^2 [n \cos(\chi/2) - 1] [n - \cos(\chi/2)]}{4 \cos(\chi/2) [n^2 + 1 - 2n \cos(\chi/2)]^2} d\phi, & n &= \sqrt{1 + \frac{2U_0}{mv_\infty^2}}, \\
0 < \chi < \chi_{max} &= 2 \arccos \frac{1}{n}, & \sigma &= \pi a^2
\end{aligned}$$