

§19 RUTHERFORD'S FORMULA

PROBLEM 1.

Find the effective cross-section for scattering in a field $U = \alpha/r^2$ ($\alpha > 0$).

SOLUTION: Let the particle of mass m come from infinity with velocity v_∞ and impact parameter ρ . It is convenient to introduce the positive length a defined by

$$a^2 = \frac{2\alpha}{mv_\infty^2}.$$

With $U(r) = \alpha/r^2$, we have $2U/(mv_\infty^2) = a^2/r^2$, so the angle between the direction of closest approach and the asymptote of the trajectory is

$$\phi_0 = \int_{r_{\min}}^{\infty} \frac{\rho \, dr}{r^2 \sqrt{1 - \frac{\rho^2}{r^2} - \frac{a^2}{r^2}}} = \int_{r_{\min}}^{\infty} \frac{\rho \, dr}{r^2 \sqrt{1 - (\rho^2 + a^2)/r^2}}.$$

The distance of closest approach is the zero of the radicand,

$$r_{\min} = \sqrt{\rho^2 + a^2}.$$

We substitute $u = 1/r$, so that $dr/r^2 = -du$. When r goes from $r_{\min}$ to ∞ , u goes from $1/\sqrt{\rho^2 + a^2}$ to 0, and the minus sign is absorbed by reversing the limits,

$$\begin{aligned} \phi_0 &= \rho \int_0^{1/\sqrt{\rho^2 + a^2}} \frac{du}{\sqrt{1 - (\rho^2 + a^2)u^2}} \\ &= \frac{\rho}{\sqrt{\rho^2 + a^2}} \left[\arcsin \left(\sqrt{\rho^2 + a^2} u \right) \right]_0^{1/\sqrt{\rho^2 + a^2}} = \frac{\pi \rho}{2\sqrt{\rho^2 + a^2}}. \end{aligned}$$

The scattering angle is therefore

$$\chi = \pi - 2\phi_0 = \pi \left(1 - \frac{\rho}{\sqrt{\rho^2 + a^2}} \right) = \pi \left[1 - \left(1 + \frac{2\alpha}{mv_\infty^2 \rho^2} \right)^{-1/2} \right].$$

Since $\rho/\sqrt{\rho^2 + a^2}$ increases from 0 to 1 as ρ goes from 0 to ∞ , χ decreases monotonically from π for a head-on collision to 0 as $\rho \rightarrow \infty$. Thus each angle $0 < \chi < \pi$ corresponds to a single impact parameter, and we can invert this relation. Writing it as

$$\frac{\rho}{\sqrt{\rho^2 + a^2}} = \frac{\pi - \chi}{\pi},$$

and squaring, we get $\pi^2 \rho^2 = (\pi - \chi)^2 (\rho^2 + a^2)$, i.e.

$$\rho^2 [\pi^2 - (\pi - \chi)^2] = a^2 (\pi - \chi)^2 \Rightarrow \rho^2 = a^2 \frac{(\pi - \chi)^2}{\chi(2\pi - \chi)},$$

where we used $\pi^2 - (\pi - \chi)^2 = \chi(2\pi - \chi)$. To differentiate, let $y = \pi - \chi$, so that $\rho^2 = a^2 y^2 / (\pi^2 - y^2)$ and

$$\frac{d(\rho^2)}{dy} = a^2 \frac{2y(\pi^2 - y^2) + 2y^3}{(\pi^2 - y^2)^2} = \frac{2\pi^2 a^2 y}{(\pi^2 - y^2)^2}.$$

Since $dy = -d\chi$,

$$\frac{d(\rho^2)}{d\chi} = -\frac{2\pi^2 a^2 (\pi - \chi)}{\chi^2 (2\pi - \chi)^2}.$$

The particles scattered into the range $d\chi$ are those passing through an annulus of radius ρ and width $|d\rho|$ in the incident beam, so

$$d\sigma = 2\pi\rho \left| \frac{d\rho}{d\chi} \right| d\chi = \pi \left| \frac{d(\rho^2)}{d\chi} \right| d\chi.$$

In terms of the element of solid angle $do = 2\pi \sin \chi \, d\chi$, this becomes

$$d\sigma = \frac{1}{2 \sin \chi} \left| \frac{d(\rho^2)}{d\chi} \right| do = \pi^2 a^2 \frac{\pi - \chi}{\chi^2 (2\pi - \chi)^2 \sin \chi} do.$$

This is a differential cross-section. Its integral over all angles diverges at $\chi = 0$, since particles with

arbitrarily large impact parameters are still deflected, by arbitrarily small angles. As a check, for $\rho \gg a$ we have $\rho/\sqrt{\rho^2 + a^2} \simeq 1 - a^2/(2\rho^2)$, hence

$$\chi \simeq \frac{\pi a^2}{2\rho^2} = \frac{\pi\alpha}{mv_\infty^2 \rho^2}, \quad \frac{d\sigma}{d\alpha} \simeq \pi^2 a^2 \frac{\pi}{4\pi^2 \chi^3} = \frac{\pi\alpha}{2mv_\infty^2 \chi^3}.$$

The same small-angle result follows directly from $\rho^2 \simeq \pi\alpha/(mv_\infty^2 \chi)$, which gives $d\sigma = \pi|d(\rho^2)/d\chi| d\chi = \pi^2 \alpha d\chi/(mv_\infty^2 \chi^2)$ with $d\alpha \simeq 2\pi\chi d\chi$. Substituting a^2 , we thus get

ANSWER TO PROBLEM 1.

$$\chi = \pi \left[1 - \left(1 + \frac{2\alpha}{mv_\infty^2 \rho^2} \right)^{-1/2} \right],$$

$$d\sigma = \frac{2\pi^2 \alpha}{mv_\infty^2} \frac{\pi - \chi}{\chi^2 (2\pi - \chi)^2} \frac{d\alpha}{\sin \chi}, \quad 0 < \chi < \pi$$