

PROBLEM 7.

Deduce the form of a scattering field $U(r)$, given the effective cross-section as a function of the angle of scattering for a given energy E . It is assumed that $U(r)$ decreases monotonically with r (a repulsive field), with $U(0) > E$ and $U(\infty) = 0$ (O. B. Firsov 1953).

SOLUTION: Let $d\sigma/d\theta$ be the effective cross-section per unit scattering angle, *i.e.* already integrated over the azimuth. In terms of the cross-section per unit solid angle, since $d\omega = 2\pi \sin \theta d\theta$,

$$\frac{d\sigma}{d\theta} = 2\pi \sin \theta \frac{d\sigma}{d\omega}.$$

As in the book, we assume that the relation between ρ and χ is one-to-one, with χ decreasing monotonically from π (head-on collision, $\rho = 0$) to 0 (no deflection, $\rho = \infty$). Note that this is an additional assumption: the monotonicity of U alone does not guarantee it. The particles scattered through angles between χ and π are then exactly those with impact parameters smaller than $\rho(\chi)$, so they fill a disc of area $\pi\rho^2$ in the incident beam, and

$$\pi\rho^2(\chi) = \int_{\chi}^{\pi} \frac{d\sigma}{d\theta} d\theta.$$

Knowing the cross-section, this integral gives $\rho(\chi)$, which we invert to get the scattering angle $\chi(\rho)$ as a function of the impact parameter. The problem is thus reduced to finding $U(r)$ from $\chi(\rho)$.

The scattering angle in a central field is given by

$$\chi = \pi - 2 \int_{r_{min}}^{\infty} \frac{\rho dr}{r^2 \sqrt{1 - \frac{\rho^2}{r^2} - \frac{U(r)}{E}}}.$$

We now introduce the variables

$$s = \frac{1}{r}, \quad x = \frac{1}{\rho^2}, \quad w(s) = \sqrt{1 - \frac{U(1/s)}{E}},$$

and write $\hat{\chi}(x) = \chi(x^{-1/2})$ for the same scattering angle expressed in terms of x . Since $dr = -ds/s^2$, we have $\rho dr/r^2 = -\rho ds$. When r goes from r_{min} to ∞ , s goes from $s_0 = 1/r_{min}$ to 0, and the minus sign is absorbed by reversing the limits. Thus

$$\frac{\pi - \hat{\chi}(x)}{2} = \int_0^{s_0} \frac{\rho ds}{\sqrt{w^2(s) - \rho^2 s^2}} = \int_0^{s_0(x)} \frac{ds}{\sqrt{xw^2(s) - s^2}},$$

where we divided the numerator and the denominator by $\rho = x^{-1/2}$. The turning point $s_0(x)$ is the zero of the radicand,

$$xw^2(s_0) - s_0^2 = 0.$$

Since U decreases with r , it increases with s , so $w(s)$ decreases and $s/w(s)$ increases with s . The radicand is therefore positive for $0 \leq s < s_0(x)$, and the condition $s < s_0(x)$ is equivalent to $x > s^2/w^2(s)$.

This is an integral equation of Abel's type, which we solve as follows. Let A be an auxiliary parameter (denoted α in the book). We divide both sides by $\sqrt{A - x}$ and integrate over x from 0 to A ,

$$\frac{1}{2} \int_0^A \frac{\pi - \hat{\chi}(x)}{\sqrt{A - x}} dx = \int_0^A \int_0^{s_0(x)} \frac{ds dx}{\sqrt{(xw^2(s) - s^2)(A - x)}}.$$

The domain of integration on the right is $0 < x < A$, $0 < s < s_0(x)$. From the previous remark, it can equally be described by $0 < s < s_0(A)$ and $s^2/w^2(s) < x < A$. Reversing the order of integration and writing $xw^2 - s^2 = w^2(x - s^2/w^2)$, we get

$$\frac{1}{2} \int_0^A \frac{\pi - \hat{\chi}(x)}{\sqrt{A - x}} dx = \int_0^{s_0(A)} \frac{ds}{w(s)} \int_{s^2/w^2(s)}^A \frac{dx}{\sqrt{(x - s^2/w^2(s))(A - x)}}.$$

For the inner integral, let $a = s^2/w^2$ and substitute $x = a + (A - a)\sin^2 \varphi$, so that $dx = 2(A -$

a) $\sin \varphi \cos \varphi \, d\varphi$ and $\sqrt{(x-a)(A-x)} = (A-a) \sin \varphi \cos \varphi$. Then

$$\int_a^A \frac{dx}{\sqrt{(x-a)(A-x)}} = \int_0^{\pi/2} 2 \, d\varphi = \pi,$$

independently of a and A . Hence

$$\frac{1}{2} \int_0^A \frac{\pi - \widehat{\chi}(x)}{\sqrt{A-x}} \, dx = \pi \int_0^{s_0(A)} \frac{ds}{w(s)}.$$

On the left, the first term gives $\frac{1}{2}\pi \int_0^A (A-x)^{-1/2} \, dx = \pi\sqrt{A}$. For the second term, we integrate by parts with $dx/\sqrt{A-x} = -d(2\sqrt{A-x})$,

$$\begin{aligned} \int_0^A \frac{\widehat{\chi}(x) \, dx}{\sqrt{A-x}} &= \left[-2\widehat{\chi}(x)\sqrt{A-x} \right]_0^A + 2 \int_0^A \widehat{\chi}'(x)\sqrt{A-x} \, dx \\ &= 2 \int_0^A \widehat{\chi}'(x)\sqrt{A-x} \, dx, \end{aligned}$$

since the boundary term vanishes at $x = A$ and $\widehat{\chi}(0) = 0$ (a particle with infinite impact parameter is not deflected). Therefore

$$\pi\sqrt{A} - \int_0^A \widehat{\chi}'(x)\sqrt{A-x} \, dx = \pi \int_0^{s_0(A)} \frac{ds}{w(s)}.$$

We differentiate this with respect to A . The integrand on the left vanishes at the upper limit $x = A$, so only the derivative of $\sqrt{A-x}$ contributes,

$$\frac{\pi}{2\sqrt{A}} - \frac{1}{2} \int_0^A \frac{\widehat{\chi}'(x) \, dx}{\sqrt{A-x}} = \frac{\pi}{w(s_0)} \frac{ds_0}{dA}.$$

This holds for every A . Since A and s_0 are related by $Aw^2(s_0) = s_0^2$, we may instead take s_0 as the independent variable, rename it s , and put $A = s^2/w^2$ with $w = w(s)$. Multiplying by $dA = 2(s/w) \, d(s/w)$ and using $\sqrt{A} = s/w$, we get

$$\pi \, d\left(\frac{s}{w}\right) - \frac{s}{w} \, d\left(\frac{s}{w}\right) \int_0^{s^2/w^2} \frac{\widehat{\chi}'(x) \, dx}{\sqrt{s^2/w^2 - x}} = \frac{\pi \, ds}{w}.$$

Since

$$\pi \, d\left(\frac{s}{w}\right) - \frac{\pi \, ds}{w} = -\frac{\pi s \, dw}{w^2} = \frac{s}{w} (-\pi \, d \log w),$$

dividing by s/w gives

$$-\pi \, d \log w = d\left(\frac{s}{w}\right) \int_0^{s^2/w^2} \frac{\widehat{\chi}'(x) \, dx}{\sqrt{s^2/w^2 - x}}.$$

Let $h = s/w$, which increases from 0 with s . At $s = 0$ (i.e. $r = \infty$), $U = 0$ and $w = 1$, so $\log w = 0$. Integrating from $s = 0$, and calling η the variable of integration for h ,

$$-\pi \log w = \int_0^h d\eta \int_0^{\eta^2} \frac{\widehat{\chi}'(x) \, dx}{\sqrt{\eta^2 - x}}.$$

The domain $0 < \eta < h$, $0 < x < \eta^2$ is also $0 < x < h^2$, $\sqrt{x} < \eta < h$. Interchanging the integrations,

$$-\pi \log w = \int_0^{h^2} \widehat{\chi}'(x) \left[\int_{\sqrt{x}}^h \frac{d\eta}{\sqrt{\eta^2 - x}} \right] dx = \int_0^{h^2} \widehat{\chi}'(x) \operatorname{arcosh} \left(\frac{h}{\sqrt{x}} \right) dx,$$

where we used $\int d\eta/\sqrt{\eta^2 - x} = \operatorname{arcosh}(\eta/\sqrt{x})$ and $\operatorname{arcosh} 1 = 0$.

We now go back to the variables r and ρ . We have $h = 1/(rw)$ and $\sqrt{x} = 1/\rho$, so $h/\sqrt{x} = \rho/(rw)$, and $\hat{\chi}'(x) dx = d\chi = (d\chi/d\rho) d\rho$. The limit $x = 0$ corresponds to $\rho = \infty$ and $x = h^2$ to $\rho = rw$. Hence

$$-\pi \log w = \int_{\infty}^{rw} \operatorname{arcosh} \left(\frac{\rho}{rw} \right) \frac{d\chi}{d\rho} d\rho = - \int_{rw}^{\infty} \operatorname{arcosh} \left(\frac{\rho}{rw} \right) \frac{d\chi}{d\rho} d\rho,$$

i.e.

$$w(r) = \exp \left\{ \frac{1}{\pi} \int_{rw(r)}^{\infty} \operatorname{arcosh} \left(\frac{\rho}{rw(r)} \right) \frac{d\chi}{d\rho} d\rho \right\}.$$

With the assumption adopted at the start, that χ decreases monotonically with ρ , we have $d\chi/d\rho < 0$. The exponent is then negative and $w < 1$, *i.e.* $U > 0$, as expected for a repulsive field.

To obtain the second form, we integrate by parts in ρ , keeping r and $w(r)$ fixed. Since

$$\frac{d}{d\rho} \operatorname{arcosh} \left(\frac{\rho}{rw} \right) = \frac{1}{\sqrt{\rho^2 - r^2 w^2}},$$

we have

$$\begin{aligned} \int_{rw}^{\infty} \operatorname{arcosh} \left(\frac{\rho}{rw} \right) \frac{d\chi}{d\rho} d\rho &= \left[\chi(\rho) \operatorname{arcosh} \left(\frac{\rho}{rw} \right) \right]_{rw}^{\infty} \\ &\quad - \int_{rw}^{\infty} \frac{\chi(\rho) d\rho}{\sqrt{\rho^2 - r^2 w^2}}. \end{aligned}$$

Two conditions on the behaviour of $\chi(\rho)$ at large ρ are needed here. **(a)** The boundary term vanishes at $\rho = rw$ because $\operatorname{arcosh} 1 = 0$. At $\rho \rightarrow \infty$, $\operatorname{arcosh}(\rho/rw) \simeq \log(2\rho/rw)$ grows logarithmically, so the boundary term vanishes there only if $\chi(\rho) \log \rho \rightarrow 0$. **(b)** The remaining integral must converge. Its integrand behaves as $\chi(\rho)/\rho$ at large ρ , so at infinity its convergence is equivalent to that of $\int_{\infty}^{\infty} \chi(\rho) d\rho/\rho$. Near the lower limit, the singularity $1/\sqrt{\rho - rw}$ is integrable. Condition (a) alone does not guarantee (b): for example, $\chi = 1/(\log \rho \log \log \rho)$ satisfies $\chi \log \rho \rightarrow 0$, but $\int_{\infty}^{\infty} \chi d\rho/\rho$ diverges. A sufficient condition for both is $\chi(\rho) = O(\rho^{-\varepsilon})$ with some $\varepsilon > 0$: then $\chi \log \rho = O(\rho^{-\varepsilon} \log \rho) \rightarrow 0$, and $\int_{\infty}^{\infty} \rho^{-1-\varepsilon} d\rho$ converges. For instance, for a field decreasing as r^{-n} at large distances, $\chi \propto \rho^{-n}$ for large ρ (see §20 Prob. 2). Under these conditions,

$$w(r) = \exp \left\{ -\frac{1}{\pi} \int_{rw(r)}^{\infty} \frac{\chi(\rho) d\rho}{\sqrt{\rho^2 - r^2 w^2(r)}} \right\}.$$

Both formulas give w only implicitly, since w also appears in the lower limit. However, the right-hand sides depend on r and w only through the product rw . They therefore give w as an explicit function of rw , from which $w(r)$ follows, and then

$$U(r) = E [1 - w^2(r)].$$

The derivation covered all s between 0 and the turning points s_0 . The smallest distance reached is that of the head-on collision ($\rho = 0$), r_{min} , for which $U(r_{min}) = E$ and $w = 0$. The potential is thus determined in the accessible region $r > r_{min}$. The scattering data give no information on the interior region $r < r_{min}$, where $U > E$. In summary, we get

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$$\begin{aligned} w(r) &= \exp \left\{ \frac{1}{\pi} \int_{rw(r)}^{\infty} \operatorname{arcosh} \left(\frac{\rho}{rw(r)} \right) \frac{d\chi}{d\rho} d\rho \right\} = \exp \left\{ -\frac{1}{\pi} \int_{rw(r)}^{\infty} \frac{\chi(\rho) d\rho}{\sqrt{\rho^2 - r^2 w^2(r)}} \right\}, \\ w(r) &= \sqrt{1 - \frac{U(r)}{E}}, \quad U(r) = E [1 - w^2(r)], \quad r > r_{min} \end{aligned}$$