

PROBLEM 6.

Determine the effective cross-section for particles of mass m_1 to strike a sphere of mass m_2 and radius R to which they are attracted in accordance with Newton's law.

SOLUTION: We consider the relative motion of the particle and the sphere, with r the distance between their centres. This is equivalent to the motion of a single particle of reduced mass

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

in the field

$$U(r) = -\frac{\gamma m_1 m_2}{r},$$

where γ is the gravitational constant. Let v_∞ be the relative velocity at infinity and ρ the impact parameter. The energy and angular momentum of the relative motion are

$$E = \frac{1}{2} m v_\infty^2, \quad M = m v_\infty \rho,$$

and the radial motion is governed by the energy equation

$$\frac{1}{2} m \dot{r}^2 + \frac{M^2}{2mr^2} - \frac{\gamma m_1 m_2}{r} = E.$$

The distance of closest approach decreases as ρ decreases, so the particle strikes the sphere if $\rho \leq \rho_{max}$, where ρ_{max} corresponds to the limiting trajectory which grazes the sphere. For this trajectory, $\dot{r} = 0$ at $r = R$. Substituting $M = m v_\infty \rho_{max}$, we have

$$\frac{m v_\infty^2 \rho_{max}^2}{2R^2} - \frac{\gamma m_1 m_2}{R} = \frac{1}{2} m v_\infty^2.$$

Multiplying by $2R^2/(m v_\infty^2)$ and using $m_1 m_2/m = m_1 + m_2$,

$$\begin{aligned} \rho_{max}^2 &= R^2 \left(1 + \frac{2\gamma m_1 m_2}{m R v_\infty^2} \right) \\ &= R^2 \left(1 + \frac{2\gamma(m_1 + m_2)}{R v_\infty^2} \right). \end{aligned}$$

The effective cross-section is obtained by summing the areas of all the annuli of the incident beam that lead to a collision with the sphere,

$$\sigma = \int_0^{\rho_{max}} 2\pi \rho \, d\rho = \pi \rho_{max}^2 = \pi R^2 \left(1 + \frac{2\gamma(m_1 + m_2)}{R v_\infty^2} \right).$$

The second term describes the gravitational focusing of the trajectories towards the sphere. In the high-speed limit $v_\infty^2 \gg \gamma(m_1 + m_2)/R$, it becomes negligible and we recover the geometric cross-section $\sigma = \pi R^2$. As in the book, we assume $m_2 \gg m_1$, so that $m_1 + m_2 \simeq m_2$. We thus get

ANSWER TO PROBLEM 6.

$$\sigma \simeq \pi R^2 \left(1 + \frac{2\gamma m_2}{R v_\infty^2} \right)$$