

## PROBLEM 5.

The same as §18 Prob. 4, but for a field  $U = -\alpha/r^n$  ( $n > 2, \alpha > 0$ ).

SOLUTION: As in §18 Prob. 4, the particle comes from infinity with velocity  $v_\infty$  and impact parameter  $\rho$ , so that

$$E = \frac{1}{2}mv_\infty^2, \quad M = mv_\infty\rho.$$

The effective potential energy is now

$$U_{\text{eff}}(r) = \frac{M^2}{2mr^2} - \frac{\alpha}{r^n}.$$

Since  $n > 2$ , the centrifugal term dominates at large  $r$  and the attractive term dominates at small  $r$ . Thus  $U_{\text{eff}} \rightarrow 0^+$  as  $r \rightarrow \infty$ ,  $U_{\text{eff}} \rightarrow -\infty$  as  $r \rightarrow 0$ , and  $U_{\text{eff}}$  has a maximum at some  $r_b$ , given by

$$U'_{\text{eff}}(r_b) = -\frac{M^2}{mr_b^3} + \frac{n\alpha}{r_b^{n+1}} = 0 \Rightarrow \frac{M^2}{m} = \frac{n\alpha}{r_b^{n-2}}.$$

This barrier analysis assumes  $M \neq 0$ . A head-on particle ( $\rho = 0$ ) has no centrifugal barrier:  $U_{\text{eff}} = -\alpha/r^n$  is monotonic, and the particle falls to the centre. The height of this barrier is

$$U_{\text{eff}}(r_b) = \frac{n\alpha}{2r_b^n} - \frac{\alpha}{r_b^n} = \frac{(n-2)\alpha}{2r_b^n}.$$

If  $E$  is smaller than the barrier height, the particle reaches a turning point outside the barrier and goes back to infinity. If  $E$  exceeds the barrier height, then  $E - U_{\text{eff}} > 0$  for all  $r$ , so the particle has no radial turning point and falls to the centre. Since the barrier height increases with  $M$ , the capture occurs for  $\rho < \rho_{\text{max}}$ , where  $\rho_{\text{max}}$  is the impact parameter for which  $E$  is equal to the barrier height. In this limiting case, the incoming particle approaches asymptotically the unstable circular orbit  $r = r_b$ . Indeed, at  $r = r_b$  both  $E - U_{\text{eff}}$  and  $U'_{\text{eff}}$  vanish, so  $r_b$  is a point of (unstable) equilibrium of the radial motion. Near it,  $\frac{1}{2}m\dot{r}^2 = E - U_{\text{eff}} \propto (r - r_b)^2$ , so the time  $\int dr/|\dot{r}| \propto \int dr/|r - r_b|$  needed to reach  $r_b$  diverges logarithmically. In contrast with §18 Prob. 4, the critical particle therefore never reaches the centre. This single value of  $\rho$  does not affect the cross-section.

Denote by  $r_c$  the position of the maximum at the critical impact parameter. The condition  $E = U_{\text{eff}}(r_c)$  gives

$$\frac{1}{2}mv_\infty^2 = \frac{(n-2)\alpha}{2r_c^n} \Rightarrow r_c^n = \frac{(n-2)\alpha}{mv_\infty^2}.$$

Using  $M = mv_\infty\rho_{\text{max}}$  in the condition for the maximum, we find

$$\begin{aligned} \rho_{\text{max}}^2 &= \frac{M^2}{m^2v_\infty^2} = \frac{n\alpha}{mv_\infty^2r_c^{n-2}} \\ &= \frac{n}{n-2} \frac{r_c^n}{r_c^{n-2}} = \frac{n}{n-2} r_c^2, \end{aligned}$$

where we used  $\alpha/(mv_\infty^2) = r_c^n/(n-2)$  in the second line. The effective cross-section is obtained by summing the areas of all the annuli of the incident beam that lead to a fall,

$$\sigma = \int_0^{\rho_{\text{max}}} 2\pi\rho d\rho = \pi\rho_{\text{max}}^2 = \frac{\pi n}{n-2} r_c^2.$$

We thus get

## ANSWER TO PROBLEM 5.

$$\sigma = \frac{\pi n}{n-2} \left( \frac{(n-2)\alpha}{mv_\infty^2} \right)^{2/n} = \frac{\pi n}{n-2} \left( \frac{(n-2)\alpha}{2E} \right)^{2/n}$$