

PROBLEM 4.

Determine the effective cross-section for a particle to “fall” to the centre of a field $U = -\alpha/r^2$.

SOLUTION: Let the particle come from infinity with velocity v_∞ and impact parameter ρ , and take $\alpha > 0$. Its energy and angular momentum are

$$E = \frac{1}{2}mv_\infty^2, \quad M = mv_\infty\rho.$$

The radial motion is governed by the energy equation

$$\frac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r) = E,$$

where the effective potential energy is

$$U_{\text{eff}}(r) = \frac{M^2}{2mr^2} - \frac{\alpha}{r^2} = \frac{1}{r^2} \left(\frac{M^2}{2m} - \alpha \right).$$

This is the field of §15 Prob. 2, where the same condition on M^2 separates the possible paths. Since both terms vary as r^{-2} , the sign of the bracket determines the behaviour of the particle near the centre. If $M^2 > 2m\alpha$, then U_{eff} is a repulsive centrifugal barrier which tends to $+\infty$ as $r \rightarrow 0$, so the particle reaches a turning point where $U_{\text{eff}}(r_{\min}) = E$ and goes back to infinity. If $M^2 < 2m\alpha$, then $U_{\text{eff}} < 0$ for all r , so $E - U_{\text{eff}} > 0$ everywhere. The radial velocity never vanishes and an incoming particle has no radial turning point: it falls to the centre. In the limiting case $M^2 = 2m\alpha$, we have $U_{\text{eff}} = 0$, thus $\dot{r} = -v_\infty$ and the particle also reaches the centre. The critical trajectory falls to the centre because the centrifugal and attractive terms have the same dependence r^{-2} . At the critical value of M they cancel identically, so there is no barrier at all and nothing can stop the particle at a finite distance. This differs from §18 Prob. 5, where the two terms have different powers of r and U_{eff} has a maximum at a finite radius.

The particle therefore falls to the centre if $M^2 \leq 2m\alpha$, i.e. if

$$m^2v_\infty^2\rho^2 \leq 2m\alpha \Rightarrow \rho \leq \rho_{\max}, \quad \rho_{\max}^2 = \frac{2\alpha}{mv_\infty^2}.$$

The effective cross-section is obtained by summing the areas of all the annuli of the incident beam that lead to a fall,

$$\sigma = \int_0^{\rho_{\max}} 2\pi\rho \, d\rho = \pi\rho_{\max}^2.$$

We thus get

ANSWER TO PROBLEM 4.

$$\sigma = \frac{2\pi\alpha}{mv_\infty^2} = \frac{\pi\alpha}{E}$$