

PROBLEM 3.

Find the effective cross-section as a function of the velocity v_∞ for particles scattered in a field $U \sim r^{-n}$.

SOLUTION: For a central potential field, the scattering angle is determined by the integral

$$\chi = \pi - 2 \int_{r_{min}}^{\infty} \frac{\rho dr}{r^2 \sqrt{1 - \frac{\rho^2}{r^2} - \frac{2U(r)}{mv_\infty^2}}}.$$

Substituting $U(r) = \alpha r^{-n}$ and define the dimensionless variable $x = \frac{r}{\rho}$, then

$$\chi = \pi - 2 \int_{x_{min}}^{\infty} \frac{dx}{x^2 \sqrt{1 - \frac{1}{x^2} - \frac{2\alpha}{mv_\infty^2 \rho^n x^n}}}.$$

Notice the dimensionless combination $\xi = \frac{2\alpha}{mv_\infty^2 \rho^n}$, the integral simplifies to

$$\chi = \pi - 2 \int_{x_{min}}^{\infty} \frac{dx}{x^2 \sqrt{1 - \frac{1}{x^2} - \frac{\xi}{x^n}}}.$$

Thus, the scattering angle depends only on ξ ,

$$\chi = f(\xi),$$

where $f(\xi)$ is a dimensionless function determined by the integral. To invert this relationship, solve for ρ in terms of χ and v_∞ ,

$$\xi = \frac{2\alpha}{mv_\infty^2 \rho^n} \Rightarrow \rho = \left(\frac{2\alpha}{mv_\infty^2 \xi} \right)^{1/n}.$$

Let $\xi = g(\chi)$, where $g(\chi)$ is a function encoding the angular dependence. Then,

$$\begin{aligned} \rho &= \left(\frac{2\alpha}{mv_\infty^2} \right)^{1/n} (g(\chi))^{-1/n} \\ \Rightarrow \rho &\sim v_\infty^{-2/n} (g(\chi))^{-1/n}. \end{aligned}$$

The differential cross-section is thus

$$\begin{aligned} d\sigma &= \frac{\rho}{\sin \chi} \left| \frac{d\rho}{d\chi} \right| d\chi \\ d\sigma &\sim v_\infty^{-4/n} \frac{F(\chi)}{\sin \chi} d\chi, \end{aligned}$$

where $F(\chi) = \left| \frac{d}{d\chi} (g(\chi))^{-1/n} \right|$. Integrate over the solid angle

$$\sigma \sim v_\infty^{-4/n} \int F(\chi) d\chi.$$

Knowing that the integral depends only on n , not v_∞ , we have

ANSWER TO PROBLEM 3.

$$\sigma \sim v_\infty^{-4/n}$$