

PROBLEM 3.

Determine the range of possible values of the angle θ between the directions of motion of the two resulting particles in the L system.

SOLUTION: The tangent of the angle $\theta = \theta_1 + \theta_2$ is

$$\tan \theta = \frac{\tan \theta_1 + \tan \theta_2}{1 - \tan \theta_1 \tan \theta_2}.$$

Using formula (16.5) of the book to get $\tan \theta_1$ and $\tan \theta_2$, and using $\theta_0 = \theta_{10} = \pi - \theta_{20}$, the last equation become

$$\tan \theta = \frac{(v_{10} + v_{20})V \sin \theta_0}{V^2 + (v_{10} - v_{20})V \cos \theta_0 - v_{10}v_{20}}$$

or equivalently

$$\cot \theta = \frac{V^2 + (v_{10} - v_{20})V \cos \theta_0 - v_{10}v_{20}}{(v_{10} + v_{20})V \sin \theta_0}.$$

In order to find the extrema, we take the derivative with respect to θ_0 ,

$$\begin{aligned} \frac{\partial \cot \theta}{\partial \theta_0} = 0 &= \frac{(v_{20} - v_{10})V^2 - V \cos \theta_0 (V^2 - v_{10}v_{20})}{(v_{10} + v_{20})V^2 \sin^2 \theta_0} \\ \Rightarrow \cos \theta_0 &= \frac{(v_{20} - v_{10})V}{V^2 - v_{10}v_{20}}. \end{aligned}$$

We know that $-1 \leq \cos \theta_0 \leq 1$, thus $v_{10} < V < v_{20}$. Moreover, we could calculate the derivative of the cotangent with respect to V and find the expression:

$$\frac{\partial \cot \theta}{\partial V} = \frac{V^2 + v_{10}v_{20}}{(v_{10} + v_{20})V^2 \sin \theta_0}$$

which is positive for all values of V .

This is why $\cot \theta$ may assume all the values between 0 and π when $v_{10} < V < v_{20}$, because $\frac{\partial \cot \theta}{\partial \theta_0} \neq 0$ and $\frac{\partial \cot \theta}{\partial V} \neq 0$ for all V, θ_0 in that interval. It is easy to see that if V is kept constant and θ_0 is varied between 0 and π , $\cot \theta$ spans the interval $(-\infty, +\infty)$; similarly, if θ_0 is kept constant and V is varied between 0 and $+\infty$, $\cot \theta$ spans the interval $(-\infty, +\infty)$.

When $V \leq v_{10}$, it is easy to see that $\cot \theta$ has a maximum value. In fact,

$$\lim_{\theta_0 \rightarrow 0^+} \cot(x) = -\infty, \quad \lim_{\theta_0 \rightarrow \pi^-} \cot(x) = -\infty$$

and the maximum value of the cotangent is reached when:

$$\cos \theta_0 = \frac{(v_{20} - v_{10})V}{V^2 - v_{10}v_{20}}$$

The calculation of the maximum value leads to:

$$\cot \theta = \frac{V^2 + (v_{10} - v_{20})V \cos \theta_0 - v_{10}v_{20}}{(v_{10} + v_{20})V \sin \theta_0} = -\frac{\sqrt{(V^2 - v_{10}v_{20})^2 - (v_{20} - v_{10})^2 V^2}}{(v_{10} + v_{20})V}$$

Then, after transforming the cotangent into a sine, one gets:

$$\sin \theta = \frac{(v_{20} + v_{10})V}{V^2 + v_{10}v_{20}}$$

If we put:

$$\alpha = \sin^{-1} \left(\frac{(v_{20} + v_{10})V}{V^2 + v_{10}v_{20}} \right)$$

then, in the interval where $V \leq v_{10}$, we have the following ranges for θ : $\pi - \alpha \leq \theta \leq \pi$. Similarly, when

$V \geq v_{20}$, it is easy to see that $\cot \theta$ has a minimum value. In fact,

$$\lim_{\theta_0 \rightarrow 0^+} \cot(x) = +\infty, \quad \lim_{\theta_0 \rightarrow \pi^-} \cot(x) = +\infty$$

and the minimum value of the cotangent is reached when $\theta = \alpha$. In this case, the ranges for θ are $0 \leq \theta \leq \alpha$ (the equality $\theta = 0$ is satisfied only when $V \rightarrow +\infty$). The final solution is thus

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$$\text{if } v_{10} < V < v_{20} \quad \Rightarrow \quad 0 < \theta < \pi$$

$$\text{if } V \leq v_{10} \quad \Rightarrow \quad \pi - \alpha \leq \theta \leq \pi$$

$$\text{if } V \geq v_{20} \quad \Rightarrow \quad 0 \leq \theta \leq \alpha$$

where

$$\alpha = \sin^{-1} \left(\frac{(v_{20} + v_{10})V}{V^2 + v_{10}v_{20}} \right).$$