

PROBLEM 3.

When a small correction $\delta U(r)$ is added to the potential energy $U = -\alpha/r$, the paths of finite motion are no longer closed, and at each revolution the perihelion is displaced through a small angle $\delta\phi$. Find $\delta\phi$ when

(a)

$$\delta U = \frac{\beta}{r^2}$$

SOLUTION: From formula (14.10) of the book, we have

$$\begin{aligned}\Delta\phi &= 2 \int_{r_{\min}}^{r_{\max}} \frac{M dr}{r^2 \sqrt{2m(E - U) - \frac{M^2}{r^2}}} \\ &= -2 \frac{\partial}{\partial M} \int_{r_{\min}}^{r_{\max}} \sqrt{2m(E - U) - \frac{M^2}{r^2}} dr \\ &= -2 \frac{\partial}{\partial M} \int_{r_{\min}}^{r_{\max}} \sqrt{2m \left(E + \frac{\alpha}{r} - \delta U \right) - \frac{M^2}{r^2}} dr.\end{aligned}$$

Expanding the integrand in powers of δU involves using a Taylor series expansion. Let's denote the integrand as F :

$$F = \sqrt{2m \left(E + \frac{\alpha}{r} + \delta U \right) - \frac{M^2}{r^2}}$$

We want to expand F around $\delta U = 0$. The expansion will look like:

$$F = F_0 + F_1 \delta U + F_2 (\delta U)^2 + \dots$$

Here, F_0 is the value of F at $\delta U = 0$, F_1 is the first derivative with respect to δU at $\delta U = 0$, and so on. Let's find the derivatives:

$$\begin{aligned}F_0 &= \sqrt{2m \left(E + \frac{\alpha}{r} \right) - \frac{M^2}{r^2}} \\ F_1 &= \left. \frac{\partial F}{\partial (\delta U)} \right|_{\delta U=0} = \frac{m}{\sqrt{2m \left(E + \frac{\alpha}{r} \right) - \frac{M^2}{r^2}}} \\ F_2 &= \left. \frac{\partial^2 F}{\partial (\delta U)^2} \right|_{\delta U=0} = -\frac{m^2 \delta U}{\left(2m \left(E + \frac{\alpha}{r} \right) - \frac{M^2}{r^2} \right)^{3/2}}.\end{aligned}$$

Therefore, the expanded expression in powers of δU is:

$$F = F_0 + \frac{m}{\sqrt{2m \left(E + \frac{\alpha}{r} \right) - \frac{M^2}{r^2}}} \delta U - \frac{m^2}{2 \left(2m \left(E + \frac{\alpha}{r} \right) - \frac{M^2}{r^2} \right)^{3/2}} \delta U^2 + \dots$$

After plugging the expanded expression of F in the integral, we see that the zero-order term gives 2π . The first-order term gives the required change $\delta\phi$:

$$\delta\phi = \frac{\partial}{\partial M} \int_{r_{\min}}^{r_{\max}} \frac{2m\delta U}{\sqrt{2m \left(E + \frac{\alpha}{r} \right) - \frac{M^2}{r^2}}}. \quad (3.11)$$

We can change the integration over r to one over ϕ , along the path of the unperturbed motion, using the substitution

$$\phi = \cos^{-1} \frac{(M/r) - (m\alpha/M)}{\sqrt{2mE + \frac{m^2\alpha^2}{M^2}}}$$

with the differential form

$$\begin{aligned}
\frac{r^2}{M} d\phi &= \frac{-1}{\sqrt{1 - \left(\frac{(M/r) - (m\alpha/M)}{\sqrt{2mE + \frac{m^2\alpha^2}{M^2}}} \right)^2}} \frac{1}{\sqrt{2mE + \frac{m^2\alpha^2}{M^2}}} \\
&= \sqrt{2m \left(E + \frac{\alpha}{r} \right) - \frac{M^2}{r^2}}.
\end{aligned}$$

Hence, the integral Eq. (3.11) become

$$\begin{aligned}
\delta\phi &= \frac{\partial}{\partial M} \left(\frac{2m}{M} \int_0^\pi r^2 \delta U d\phi \right) \\
&= \frac{\partial}{\partial M} \left(\frac{2m}{M} \int_0^\pi r^2 \frac{\beta}{r^2} d\phi \right) \\
&= \frac{\partial}{\partial M} \left(\frac{2\pi m\beta}{M} \right) \\
&= -\frac{2\pi m\beta}{M^2}.
\end{aligned} \tag{3.12}$$

Using formula (15.4) of the book we can finally write

ANSWER TO PROBLEM 3 (A).

$$\delta\phi = -\frac{2\pi\beta}{\alpha p}$$

(b)

$$\delta U = \frac{\gamma}{r^3}$$

SOLUTION: From Eq. (3.12), we have

$$\delta\phi = \frac{\partial}{\partial M} \left(\frac{2m}{M} \int_0^\pi \frac{\gamma}{r} d\phi \right).$$

Using formula (15.5) of the book, the integral become

$$\begin{aligned}
\delta\phi &= \frac{\partial}{\partial M} \left(\frac{2m}{M} \int_0^\pi \frac{\gamma}{p} (1 + e \cos \phi) d\phi \right) \\
&= \frac{\partial}{\partial M} \left(\frac{2\pi\gamma m}{Mp} \right) \\
&= -\frac{6\pi\alpha\gamma m^2}{M^4}.
\end{aligned}$$

The last expression can also be written as

ANSWER TO PROBLEM 3 (B).

$$\delta\phi = -\frac{6\pi\gamma}{\alpha p^2}$$