

PROBLEM 2.

Integrate the equations of motion for a particle in a central field

$$U = -\frac{\alpha}{r^2} \quad (\alpha > 0).$$

SOLUTION: From formula (14.6) of the book, we have the integral

$$\begin{aligned} t &= \int \frac{dr}{\sqrt{\frac{2}{m}(E - U(r)) - \frac{M^2}{m^2 r^2}}} \\ &= \int \frac{dr}{\sqrt{\frac{2E}{m} r^2 + \frac{2\alpha}{m} - \frac{M^2}{m^2}}} \\ &= \sqrt{\frac{m}{2E}} \int \frac{r dr}{\sqrt{r^2 + \left(\frac{\alpha}{E} - \frac{M^2}{2mE}\right)}}. \end{aligned}$$

We use the substitution

$$u = r^2 + \left(\frac{\alpha}{E} - \frac{M^2}{2mE}\right)$$

with the differential form

$$du = 2r dr.$$

Hence, the integral become

$$\begin{aligned} t &= \frac{1}{2} \sqrt{\frac{m}{2E}} \int \frac{du}{\sqrt{u}} \\ &= \sqrt{\frac{m}{2E}} \sqrt{u} \\ &= \sqrt{\frac{m}{2E}} \sqrt{r^2 + \frac{\alpha}{E} - \frac{M^2}{2mE}} \\ &= \frac{1}{E} \sqrt{\frac{m}{2}} \left(Er^2 + \alpha - \frac{M^2}{2m} \right). \end{aligned}$$

Formula (14.7) of the book gives us the equation of the path

$$\begin{aligned} \phi &= \int \frac{M dr}{r^2 \sqrt{2m(E - U(r)) - \frac{M^2}{r^2}}} \\ &= \int \frac{M dr}{r^2 \sqrt{2mE + \frac{2m\alpha}{r^2} - \frac{M^2}{r^2}}}. \end{aligned}$$

Using the substitution

$$u = \frac{1}{r}$$

with the differential form

$$du = -\frac{1}{r^2} dr,$$

the integral become

$$\begin{aligned} \phi &= - \int \frac{du}{\sqrt{2mE + (2m\alpha - M^2)u^2}} \\ &= -\frac{1}{\sqrt{2mE}} \int \frac{du}{\sqrt{1 + (ku)^2}}, \end{aligned} \tag{3.9}$$

with

$$k = \sqrt{\frac{2m\alpha - M^2}{2mE}}. \tag{3.10}$$

From there, the solution must be divided for the three possible cases : **(a)** $E > 0$, $M^2 > 2m\alpha$; **(b)** $E > 0$,

$M^2 < 2m\alpha$; (c) $E < 0$ $M^2 < 2m\alpha$. It is also interesting to know that the path is a Cotes's spiral¹.

(b) : Eq. (3.10) is still

$$k = \sqrt{\frac{2m\alpha - M^2}{2mE}}.$$

By using the substitution

$$ku = \sinh \theta$$

with the differential form

$$kdu = \cosh \theta d\theta,$$

the integral Eq. (3.9) become

$$\begin{aligned} \phi &= -\frac{1}{\sqrt{2mE}} \int \frac{\cosh \theta d\theta}{k\sqrt{1 + \sinh^2 \theta}} \\ &= -\frac{1}{\sqrt{2m\alpha - M^2}} \theta \\ &= -\frac{1}{\sqrt{2m\alpha - M^2}} \sinh^{-1} \frac{k}{r}. \end{aligned}$$

The equation of the path for the case (b) is thus

$$\begin{aligned} \frac{1}{r} &= \frac{1}{k} \sinh \left(\phi \sqrt{2m\alpha - M^2} \right) \\ &= \sqrt{\frac{2mE}{2m\alpha - M^2}} \sinh \left(\phi \sqrt{\frac{2m\alpha}{M^2} - 1} \right). \end{aligned}$$

(c) : Eq. (3.10) become

$$k = \frac{\sqrt{2m\alpha - M^2}}{i\sqrt{2m|E|}}.$$

By using the substitution

$$iku = \cosh \theta$$

with the differential form

$$ikdu = \sinh \theta d\theta,$$

the integral Eq. (3.9) become

$$\begin{aligned} \phi &= -\frac{1}{i\sqrt{2m|E|}} \int \frac{\sinh \theta d\theta}{ik\sqrt{1 - \cosh^2 \theta}} \\ &= \frac{1}{\sqrt{2m\alpha - M^2}} \theta \\ &= \frac{1}{\sqrt{2m\alpha - M^2}} \cosh^{-1} \frac{ik}{r}. \end{aligned}$$

The equation of the path for the case (c) is thus

$$\begin{aligned} \frac{1}{r} &= \frac{1}{ik} \cosh \left(\phi \sqrt{2m\alpha - M^2} \right) \\ &= \sqrt{\frac{2m|E|}{2m\alpha - M^2}} \cosh \left(\phi \sqrt{\frac{2m\alpha}{M^2} - 1} \right). \end{aligned}$$

(a) : Eq. (3.10) become

¹Whittaker ET (1937). A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, with an Introduction to the Problem of Three Bodies (4th ed.). New York: Dover Publications. pp. 80–83. ISBN 978-0-521-35883-5.

$$k = \frac{i\sqrt{2m\alpha - M^2}}{\sqrt{2mE}}.$$

By using the substitution

$$ku = i \cosh \theta$$

with the differential form

$$kdu = i \sinh \theta d\theta,$$

the integral Eq. (3.9) become

$$\begin{aligned} \phi &= -\frac{1}{\sqrt{2mE}} \int \frac{i \sinh \theta d\theta}{k \sqrt{1 - \cosh^2 \theta}} \\ &= -\frac{i}{\sqrt{2m\alpha - M^2}} \theta \\ &= -\frac{i}{\sqrt{2m\alpha - M^2}} \cosh^{-1} \frac{k}{ir}. \end{aligned}$$

The equation of the path for the case (a) is thus

$$\begin{aligned} \frac{1}{r} &= \frac{i}{k} \cosh \left(\frac{\phi}{i} \sqrt{2m\alpha - M^2} \right) \\ &= \sqrt{\frac{2mE}{M^2 - 2m\alpha}} \cos \left(\phi \sqrt{1 - \frac{2m\alpha}{M^2}} \right). \end{aligned}$$

To summarize the equations of motion for a particle in a central inverse-cube law force is

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(a) for $E > 0$ and $\frac{M^2}{2m} > \alpha$,

$$\frac{1}{r} = \sqrt{\frac{2mE}{M^2 - 2m\alpha}} \cos \left(\phi \sqrt{1 - \frac{2m\alpha}{M^2}} \right)$$

(b) for $E > 0$ and $\frac{M^2}{2m} < \alpha$,

$$\frac{1}{r} = \sqrt{\frac{2mE}{2m\alpha - M^2}} \sinh \left(\phi \sqrt{\frac{2m\alpha}{M^2} - 1} \right)$$

(c) for $E < 0$ and $\frac{M^2}{2m} < \alpha$,

$$\frac{1}{r} = \sqrt{\frac{2m|E|}{2m\alpha - M^2}} \cosh \left(\phi \sqrt{\frac{2m\alpha}{M^2} - 1} \right)$$

In all three cases

$$t = \frac{1}{E} \sqrt{\frac{m}{2}} \left(Er^2 + \alpha - \frac{M^2}{2m} \right)$$