

PROBLEM 3.

Integrate the equations of motion for a pendulum of mass m_2 , with a mass m_1 at the point of support which can move on a horizontal line lying in the plane in which m_2 moves (see the figure in §5 Prob. 2).

SOLUTION: From §5 Prob. 2, the Lagrangian of the system is

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi) + m_2gl\cos\phi.$$

The co-ordinate x is cyclic, thus

$$p_x = \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x} + m_2l\dot{\phi}\cos\phi = \text{constant} \quad (3.6)$$

is conserved. It is always possible to find an inertial frame of reference where $p_x = 0$, using this frame we get

$$\begin{aligned} (m_1 + m_2)\dot{x} + m_2l\dot{\phi}\cos\phi &= 0 \\ \Rightarrow \int \left((m_1 + m_2)\dot{x} + m_2l\dot{\phi}\cos\phi \right) dt &= \text{constant} \\ \Rightarrow (m_1 + m_2)x + m_2l\sin\phi &= (m_1 + m_2)R_x = \text{constant}. \end{aligned} \quad (3.7)$$

This express the fact that the centre of mass $\mathbf{R}$ of the system does not move horizontally. It is also possible to rewrite Eq. (3.6) as

$$\begin{aligned} (m_1 + m_2)\dot{x} + m_2l\dot{\phi}\cos\phi &= 0 \\ \Rightarrow \dot{x} &= \frac{-m_2l\dot{\phi}\cos\phi}{m_1 + m_2} \end{aligned}$$

Plugging this into the energy of the system we get

$$\begin{aligned} E &= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi) - m_2gl\cos\phi \\ &= \frac{m_2^2l^2\dot{\phi}^2\cos^2\phi}{2(m_1 + m_2)} + \frac{1}{2}m_2l^2\dot{\phi}^2 - \frac{m_2^2l^2\dot{\phi}^2\cos^2\phi}{m_1 + m_2} - m_2gl\cos\phi \\ &= \frac{1}{2}m_2l^2\dot{\phi}^2 \left(1 - \frac{m_2}{m_1 + m_2}\cos^2\phi \right) - m_2gl\cos\phi. \end{aligned}$$

Hence,

$$\begin{aligned} t &= l\sqrt{\frac{m_2}{2}} \int \sqrt{\frac{1 - \frac{m_2}{m_1 + m_2}\cos^2\phi}{E + m_2gl\cos\phi}} d\phi \\ &= l\sqrt{\frac{m_2}{2(m_1 + m_2)}} \int \sqrt{\frac{m_1 + m_2\sin^2\phi}{E + m_2gl\cos\phi}} d\phi. \end{aligned}$$

Using Eq. (3.7), we can express the position of the mass m_2 as

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x + l\sin\phi \\ l\cos\phi \end{bmatrix} = \begin{bmatrix} R_x - l\sin\phi \left(1 - \frac{m_2}{m_1 + m_2} \right) \\ l\cos\phi \end{bmatrix}.$$

The path of the particle of mass m_2 is thus an arc of an ellipse center at $(R_x, 0)$ with horizontal semi-axis $lm_1/(m_1 + m_2)$ and vertical semi-axis l . It is possible to see that when $m_1 \rightarrow \infty$, the path return to the simple pendulum. The solution of the equations of motion is thus

ANSWER TO PROBLEM 3.

$$t = l\sqrt{\frac{m_2}{2}} \int \sqrt{\frac{1 - \frac{m_2}{m_1 + m_2}\cos^2\phi}{E + m_2gl\cos\phi}} d\phi \quad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} R_x - l\sin\phi \left(1 - \frac{m_2}{m_1 + m_2} \right) \\ l\cos\phi \end{bmatrix}$$