

PROBLEM 2.

Integrate the equations of motion for a particle moving on the surface of a cone (of vertical angle 2α) placed vertically and with vertex downwards in a gravitational field.

SOLUTION: We proceed as in §14 Prob. 1. By using the spherical co-ordinate, we can write the position of the particle as

$$\begin{aligned}x &= r \cos \phi \sin \alpha \\y &= r \sin \phi \sin \alpha \\z &= r \cos \alpha.\end{aligned}$$

By taking the time derivative of those, we obtain

$$\begin{aligned}\dot{x} &= \dot{r} \cos \phi \sin \alpha - r \dot{\phi} \sin \phi \sin \alpha \\ \dot{y} &= \dot{r} \sin \phi \sin \alpha + r \dot{\phi} \cos \phi \sin \alpha \\ \dot{z} &= \dot{r} \cos \alpha.\end{aligned}$$

The Lagrangian is thus

$$\begin{aligned}L &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \\ &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 \sin^2 \alpha) - mgr \cos \alpha.\end{aligned}$$

The co-ordinate ϕ is cyclic, thus

$$M_z = p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2 \dot{\phi} \sin^2 \alpha \quad (3.4)$$

is conserved. This is the z -component of the angular momentum found in §9 Prob. 1, with the distance $r \sin \alpha$ from the axis. The energy of the particle is

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 \sin^2 \alpha) + mgr \cos \alpha. \quad (3.5)$$

We can rewrite Eq. (3.4) as

$$\dot{\phi}^2 = \frac{M_z^2}{m^2 r^4 \sin^4 \alpha}$$

and substituting in Eq. (3.5), we get

$$\begin{aligned}E &= \frac{1}{2}m\dot{r}^2 + \frac{M_z^2}{2mr^2 \sin^2 \alpha} + mgr \cos \alpha \\ &= \frac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r).\end{aligned}$$

Hence,

$$t = \sqrt{\frac{m}{2}} \int \frac{dr}{\sqrt{E - U_{\text{eff}}(r)}}.$$

From Eq. (3.4), it is possible to write

$$\begin{aligned}\frac{d\phi}{dt} &= \frac{M_z}{mr^2 \sin^2 \alpha} \\ \Rightarrow \frac{d\phi}{dr} \frac{dr}{dt} &= \frac{M_z}{mr^2 \sin^2 \alpha} \\ \Rightarrow \frac{d\phi}{dr} \sqrt{\frac{2(E - U_{\text{eff}}(r))}{m}} &= \frac{M_z}{mr^2 \sin^2 \alpha} \\ \Rightarrow \phi &= \frac{M_z}{\sqrt{2m} \sin^2 \alpha} \int \frac{dr}{r^2 \sqrt{E - U_{\text{eff}}(r)}}.\end{aligned}$$

It is possible to find the range of r . At the maximum and minimum value of r , the particle has no kinetic

energy, thus

$$\begin{aligned} E &= U_{\text{eff}} \\ &= \frac{M_z^2}{2mr^2 \sin^2 \alpha} + mgr \cos \alpha \end{aligned}$$

We can find a cubic algebraic equation for r , that is

$$\begin{aligned} 0 &= r^2 (E - mgr \cos \alpha) - \frac{M_z^2}{2m \sin^2 \alpha} \\ &= mgr^3 \cos \alpha - Er^2 + \frac{M_z^2}{2m \sin^2 \alpha}. \end{aligned}$$

This equation has two positive roots, r_p and r_a , which are the turning points of the motion. To summarize, the solution of the equation of motion for a particle moving on the surface of a cone is

ANSWER TO PROBLEM 2.

$$t = \sqrt{\frac{m}{2}} \int_{r_p}^{r_a} \frac{dr}{\sqrt{E - U_{\text{eff}}(r)}} \quad \phi = \frac{M_z}{\sqrt{2m} \sin^2 \alpha} \int_{r_p}^{r_a} \frac{dr}{r^2 \sqrt{E - U_{\text{eff}}(r)}}$$