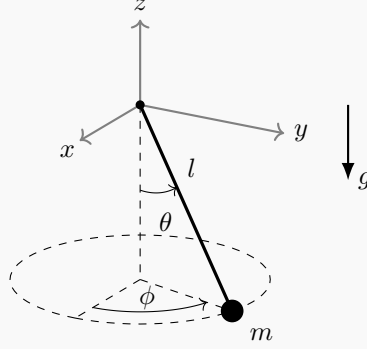


§14 MOTION IN A CENTRAL FIELD

PROBLEM 1.

Integrate the equations of motion for a spherical pendulum (a particle of mass m moving on the surface of a sphere of radius l in a gravitational field).



SOLUTION: The Cartesian position of the particle m is

$$\begin{aligned} x &= l \sin \theta \cos \phi \\ y &= l \sin \theta \sin \phi \\ z &= -l \cos \theta. \end{aligned}$$

The time derivatives of these coordinates are

$$\begin{aligned} \dot{x} &= l\dot{\theta} \cos \theta \cos \phi - l\dot{\phi} \sin \theta \sin \phi \\ \dot{y} &= l\dot{\theta} \cos \theta \sin \phi + l\dot{\phi} \sin \theta \cos \phi \\ \dot{z} &= l\dot{\theta} \sin \theta. \end{aligned}$$

Substituting these expressions in the Lagrangian, we obtain the desired Lagrangian form, that is

$$\begin{aligned} L &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \\ &= \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + mgl \cos \theta. \end{aligned}$$

The Lagrangian does not involve ϕ explicitly, thus this co-ordinate is cyclic and the generalized momentum p_ϕ is conserved. This momentum is the same as the z -component of angular momentum M_z (see §9 Prob. 2 with $r = l$) and is written

$$M_z = p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ml^2 \dot{\phi} \sin^2 \theta. \quad (3.3)$$

The energy of the pendulum is

$$E = \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) - mgl \cos \theta.$$

Substituting $\dot{\phi}$ with Eq. (3.3) in the energy we get

$$\begin{aligned} E &= \frac{1}{2}ml^2\dot{\theta}^2 + \frac{M_z^2}{2ml^2 \sin^2 \theta} - mgl \cos \theta \\ &= \frac{1}{2}ml^2\dot{\theta}^2 + U_{\text{eff}}(\theta), \end{aligned}$$

where $U_{\text{eff}}(\theta)$ is the effective potential energy. Notice that the energy was re-written as a function of only

one co-ordinate (*i.e.* θ). This is equivalent to the one particle problem we have already see, thus

$$\begin{aligned}\frac{d\theta}{dt} &= \sqrt{\frac{2(E - U_{\text{eff}}(\theta))}{ml^2}} \\ \Rightarrow t &= \int \frac{d\theta}{\sqrt{2(E - U_{\text{eff}}(\theta))}}.\end{aligned}$$

This integral lead to an elliptic integral of the first kind see §11 Prob. 1. We also need to find the solution to the angle ϕ . To do so, we can use the Eq. (3.3) with the chain rule from calculus, that is

$$\begin{aligned}\frac{M_z}{ml^2 \sin^2 \theta} &= \frac{d\phi}{dt} \\ &= \frac{d\phi}{d\theta} \frac{d\theta}{dt} \\ &= \frac{d\phi}{d\theta} \sqrt{\frac{2(E - U_{\text{eff}}(\theta))}{ml^2}}.\end{aligned}$$

Finally, we get

$$\begin{aligned}\frac{d\phi}{d\theta} &= \frac{M_z}{l \sin^2 \theta} \sqrt{\frac{1}{2m(E - U_{\text{eff}}(\theta))}} \\ \Rightarrow \phi &= \frac{M_z}{l} \sqrt{\frac{1}{2m}} \int \frac{d\theta}{\sin^2 \theta (E - U_{\text{eff}}(\theta))}.\end{aligned}$$

This integral lead to an elliptic integral of the third kind (I really don't want to solve this). It is possible to find the range of angle θ . At the maximum and minimum value of θ , the pendulum as no kinetic energy, thus

$$\begin{aligned}E &= U_{\text{eff}} \\ &= \frac{M_z^2}{2ml^2 \sin^2 \theta} - mgl \cos \theta \\ &= \frac{M_z^2}{2ml^2 (1 - \cos^2 \theta)} - mgl \cos \theta\end{aligned}$$

We can find a cubic algebraic equation for $\cos \theta$, that is

$$\begin{aligned}0 &= (E + mgl \cos \theta) (2ml^2 (1 - \cos^2 \theta)) - M_z^2 \\ &= mgl (\cos^3 \theta - \cos \theta) + E (\cos^2 \theta - 1) + \frac{M_z^2}{2ml^2}.\end{aligned}$$

The centrifugal part of the effective potential U_{eff} , namely

$$\frac{M_z^2}{2ml^2 \sin^2 \theta},$$

must be positive. As we can see, this term diverge at $\theta \rightarrow 0$ and $\theta \rightarrow \pi$, thus the angle θ is bound. This can be write as

$$\theta \in [\theta_p, \theta_a] \qquad 0 < \theta_p \leq \theta_a < \pi,$$

where the subscripts of the angles θ_p and θ_a are for the perigee and apogee. This mean that the two roots of the cubic algebraic equation for $\cos \theta$ is bound between -1 and $+1$. To summarize, the solution of the equation of motion of the spherical pendulum is

ANSWER TO PROBLEM 1.

$$t = \int_{\theta_p}^{\theta_a} \frac{d\theta}{\sqrt{2(E - U_{\text{eff}}(\theta))}} \qquad \phi = \frac{M_z}{l} \sqrt{\frac{1}{2m}} \int_{\theta_p}^{\theta_a} \frac{d\theta}{\sin^2 \theta (E - U_{\text{eff}}(\theta))}$$