

PROBLEM 2.

Determine the period of oscillation, as a function of the energy, when a particle of mass m moves in fields for which the potential energy is

(a)

$$U = A|x|^n$$

SOLUTION: The total energy of the particle is

$$E = \frac{1}{2}m\dot{x}^2 + U(x).$$

Knowing that the maximum value of E is at x_1 , this position is

$$\begin{aligned} E &= U(x_1) = A|x_1|^n \\ \Rightarrow |x_1| &= \left(\frac{E}{A}\right)^{1/n}, \\ \Rightarrow x_1 &= \pm \left(\frac{E}{A}\right)^{1/n}, \end{aligned}$$

we get the period of oscillation,

$$\begin{aligned} T &= 2\sqrt{\frac{m}{2}} \int_{-\left(\frac{E}{A}\right)^{1/n}}^{\left(\frac{E}{A}\right)^{1/n}} \frac{dx}{\sqrt{E - A|x|^n}} \\ &= 2\sqrt{2m} \int_0^{\left(\frac{E}{A}\right)^{1/n}} \frac{dx}{\sqrt{E - A|x|^n}}. \end{aligned}$$

It is possible to remove the absolute value, because the integral is over positive x , thus

$$\begin{aligned} T &= 2\sqrt{2m} \int_0^{\left(\frac{E}{A}\right)^{1/n}} \frac{dx}{\sqrt{E - Ax^n}} \\ &= 2\sqrt{\frac{2m}{E}} \int_0^{\left(\frac{E}{A}\right)^{1/n}} \frac{dx}{\sqrt{1 - \frac{A}{E}x^n}}. \end{aligned}$$

Using the substitution $y = \left(\frac{A}{E}\right)^{\frac{1}{n}} x$, we get

$$\begin{aligned} T &= 2\sqrt{\frac{2m}{E}} \int_0^1 \frac{\left(\frac{A}{E}\right)^{\frac{1}{n}} dy}{\sqrt{1 - \frac{A}{E} \left(\left(\frac{A}{E}\right)^{-\frac{1}{n}} y\right)^n}} \\ &= 2\sqrt{\frac{2m}{E}} \left(\frac{E}{A}\right)^{\frac{1}{n}} \int_0^1 \frac{dy}{\sqrt{1 - y^n}}. \end{aligned}$$

Using another substitution $u = y^n$, we get

$$T = 2\sqrt{\frac{2m}{E}} \left(\frac{E}{A}\right)^{\frac{1}{n}} \int_0^1 \frac{u^{\frac{1}{n}} du}{nu\sqrt{1 - u}}.$$

It is possible to express this integral in term of the beta function

$$\mathcal{B}(z_1, z_2) = \int_0^1 t^{z_1-1} (1-t)^{z_2-1} dt = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)},$$

with $z_1 = \frac{1}{n}$ and $z_2 = \frac{1}{2}$. This give us

$$T = \frac{2}{n} \sqrt{\frac{2m}{E}} \left(\frac{E}{A} \right)^{\frac{1}{n}} \frac{\Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{n}\right)}.$$

Knowing that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, we finally have

ANSWER TO PROBLEM 2 (A).

$$T = \frac{2}{n} \sqrt{\frac{2\pi m}{E}} \left(\frac{E}{A} \right)^{\frac{1}{n}} \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{n}\right)}$$

(b)

$$U = \frac{-U_0}{\cosh^2 \alpha x}, \quad -U_0 < E < 0$$

SOLUTION: The two boundaries of the energy are at $E = 0 \Rightarrow x = x_1$ and $E = -U_0 \Rightarrow x = 0$. The period of oscillation is thus

$$\begin{aligned} T &= 2\sqrt{2m} \int_0^{x_1} \frac{dx}{\sqrt{E - U}} \\ &= 2\sqrt{2m} \int_0^{x_1} \frac{dx}{\sqrt{E + \frac{U_0}{\cosh^2 \alpha x}}} \\ &= 2\sqrt{2m} \int_0^{x_1} \frac{\cosh^2 \alpha x \, dx}{\sqrt{E \cosh^2 \alpha x + U_0}}. \end{aligned}$$

Using the identity $\cosh^2 \alpha x = 1 + \sinh^2 \alpha x$, we get

$$T = 2\sqrt{2m} \int_0^{x_1} \frac{\cosh^2 \alpha x \, dx}{\sqrt{E(1 + \sinh^2 \alpha x) + U_0}}.$$

The substitution $y = \sinh \alpha x$, give us $dy = \alpha \cosh \alpha x dx$ and

$$\begin{aligned} y(x_1) &= \sinh \alpha x \\ &= \sqrt{\sinh^2 \alpha x} \\ &= \sqrt{\cosh^2 \alpha x - 1} \\ &= \sqrt{\frac{-U_0}{E} - 1}. \end{aligned}$$

Using this substitution, we get

$$\begin{aligned} T &= \frac{2}{\alpha} \sqrt{2m} \int_0^{\sqrt{\frac{-U_0}{E} - 1}} \frac{dy}{E(1 + y^2) + U_0} \\ &= \frac{2}{\alpha} \sqrt{2m} \int_0^{\sqrt{\frac{-U_0}{E} - 1}} \frac{dy}{E + Ey^2 - E \cosh^2 \alpha x} \end{aligned}$$

Factoring $|E|$ out of the square root (because $E < 0$), we have

$$\begin{aligned} T &= \frac{2}{\alpha} \sqrt{\frac{2m}{|E|}} \int_0^{\sqrt{\frac{-U_0}{E} - 1}} \frac{dy}{1 - \cosh^2 \alpha x + y^2} \\ &= \frac{2}{\alpha} \sqrt{\frac{2m}{|E|}} \int_0^{\sqrt{\frac{-U_0}{E} - 1}} \frac{dy}{\sqrt{\frac{-U_0}{E} - 1 + y^2}} \end{aligned}$$

This integral is of the form

$$\int_0^a \frac{dz}{\sqrt{a+z^2}} = \sin^{-1} \left(\frac{z}{a} \right) = \frac{\pi}{2}.$$

Thus, the period of oscillation is

ANSWER TO PROBLEM 2 (B).

$$T = \frac{\pi}{\alpha} \sqrt{\frac{2m}{|E|}}$$

(c)

$$U = U_0 \tan^2 \alpha x$$

SOLUTION: The period of oscillation is

$$\begin{aligned} T &= 2\sqrt{2m} \int_0^{x_1} \frac{dx}{\sqrt{E-U}} \\ &= 2\sqrt{2m} \int_0^{x_1} \frac{dx}{\sqrt{E-U_0 \tan^2 \alpha x}} \\ &= 2\sqrt{2m} \int_0^{x_1} \frac{\cos \alpha x \, dx}{\sqrt{E \cos^2 \alpha x - U_0 \sin^2 \alpha x}} \\ &= 2\sqrt{2m} \int_0^{x_1} \frac{\cos \alpha x \, dx}{\sqrt{E - (E+U_0) \sin^2 \alpha x}}. \end{aligned}$$

Using the substitution $y = i \sin \alpha x$ ($dy = i\alpha \cos \alpha x$), we get

$$\begin{aligned} T &= \frac{2\sqrt{2m}}{i\alpha} \int_0^{i \sin \alpha x_1} \frac{dy}{\sqrt{E + (E+U_0)y^2}} \\ &= \frac{2}{i\alpha} \sqrt{\frac{2m}{E+U_0}} \int_0^{i \sin \alpha x_1} \frac{dy}{\sqrt{\frac{E}{(E+U_0)} + y^2}} \\ &= \frac{2}{i\alpha} \sqrt{\frac{2m}{E+U_0}} \sin^{-1} \left(\frac{i \sin \alpha x_1}{\sqrt{\frac{E}{(E+U_0)}}} \right) \end{aligned}$$

Knowing

$$x_1 = \frac{1}{\alpha} \tan^{-1} \sqrt{\frac{E}{U_0}},$$

we finally get

$$\begin{aligned}
T &= \frac{2}{i\alpha} \sqrt{\frac{2m}{E+U_0}} \sin^{-1} \left(\frac{i \sin \left(\tan^{-1} \sqrt{\frac{E}{U_0}} \right)}{\sqrt{\frac{E}{(E+U_0)}}} \right) \\
&= \frac{2}{i\alpha} \sqrt{\frac{2m}{E+U_0}} \sin^{-1} \left(\frac{i \sqrt{\frac{E}{U_0}}}{\sqrt{1 + \frac{E}{U_0} \sqrt{\frac{E}{(E+U_0)}}}} \right) \\
&= \frac{2}{i\alpha} \sqrt{\frac{2m}{E+U_0}} \sin^{-1} i \\
&= \frac{2}{i\alpha} \sqrt{\frac{2m}{E+U_0}} \frac{\pi i}{2}.
\end{aligned}$$

Thus, the period of oscillation is

ANSWER TO PROBLEM 2 (c).

$$T = \frac{\pi}{\alpha} \sqrt{\frac{2m}{E+U_0}}$$