

§11 MOTION IN ONE DIMENSION

PROBLEM 1.

Determine the period of oscillations of a simple pendulum (a particle of mass m suspended by a string of length l in a gravitational field) as a function of the amplitude of the oscillations.

SOLUTION: The potential energy of a simple pendulum is

$$U(\phi) = -mgl \cos \phi,$$

see §5 Prob. 1. If we define ϕ_0 as the maximum value of ϕ , the potential energy is equal to the total energy at this point, that is

$$U(\phi_0) = E = -mgl \cos \phi_0.$$

The energy of the pendulum could also be write using the kinetic energy,

$$E = \frac{1}{2}ml^2\dot{\phi}^2 + U(\phi).$$

This first order differential equation can be integrate to give the period of oscillation

$$\begin{aligned} T &= 2\sqrt{\frac{ml^2}{2}} \int_{-\phi_0}^{\phi_0} \frac{d\phi}{\sqrt{E - U(\phi)}} \\ &= 4\sqrt{\frac{ml^2}{2}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{mgl \cos \phi - mgl \cos \phi_0}} \\ &= 4\sqrt{\frac{l}{2g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\cos \phi - \cos \phi_0}}. \end{aligned}$$

To solve this integral, we need first to use a trigonometric identity (I'll let you find or derive this identity), giving us

$$T = 2\sqrt{\frac{l}{g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\sin^2 \frac{\phi_0}{2} - \sin^2 \frac{\phi}{2}}}.$$

Next, we use the substitution

$$\sin \xi = \frac{\sin \frac{\phi}{2}}{\sin \frac{\phi_0}{2}}.$$

It's differential form is

$$\begin{aligned} \frac{d\phi}{d\xi} &= \frac{d}{d\xi} \left(2 \arcsin \left(\sin \frac{\phi_0}{2} \sin \xi \right) \right) \\ &= \frac{2 \sin \frac{\phi_0}{2} \cos \xi}{\sqrt{1 - \sin^2 \frac{\phi_0}{2} \sin^2 \xi}} \end{aligned}$$

The substitution, finally, give

$$\begin{aligned}
T &= 2\sqrt{\frac{l}{g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\sin^2 \frac{\phi_0}{2} - \sin^2 \xi \sin^2 \frac{\phi_0}{2}}} \\
&= 2\sqrt{\frac{l}{g}} \int_0^{\phi_0} \frac{d\phi}{\sin \frac{\phi_0}{2} \sqrt{1 - \sin^2 \xi}} \\
&= 2\sqrt{\frac{l}{g}} \int_0^{\phi_0} \frac{d\phi}{\sin \frac{\phi_0}{2} \cos \xi} \\
&= 2\sqrt{\frac{l}{g}} \int_{\arcsin(0)}^{\arcsin(1)} \frac{2 \sin \frac{\phi_0}{2} \cos \xi d\xi}{\sin \frac{\phi_0}{2} \cos \xi \sqrt{1 - \sin^2 \frac{\phi_0}{2} \sin^2 \xi}} \\
&= 4\sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} \frac{d\xi}{\sqrt{1 - \sin^2 \frac{\phi_0}{2} \sin^2 \xi}}.
\end{aligned}$$

We can also write

$$T = 4\sqrt{\frac{l}{g}} K\left(\sin \frac{\phi_0}{2}\right),$$

where

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\xi}{\sqrt{1 - k^2 \sin^2 \xi}}. \quad (3.1)$$

The integral K is known as the complete elliptic integral of the first kind. To solve this integral, we first need to see that the integrand is of the form

$$f(x) = (1 - x)^{-\frac{1}{2}}.$$

The n derivative of this function is

$$f^{(n)}(x) = \frac{(2n-1)!!}{2^n} (1-x)^{-\frac{2n+1}{2}},$$

thus the Maclaurin Series is

$$f(x) = \sum_{n \geq 0} \frac{(2n-1)!!}{2^n \cdot n!} x^n.$$

Using this result in Eq. (3.1), we get

$$\begin{aligned}
K(k) &= \int_0^{\frac{\pi}{2}} \sum_{n \geq 0} \frac{(2n-1)!!}{2^n \cdot n!} k^{2n} \sin^{2n} \xi d\xi \\
&= \sum_{n \geq 0} \frac{(2n-1)!!}{2^n \cdot n!} k^{2n} \int_0^{\frac{\pi}{2}} \sin^{2n} \xi d\xi.
\end{aligned}$$

This last integral can be solve using the beta function

$$\mathcal{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2 \int_0^{\frac{\pi}{2}} \sin^{2x-1} t \cos^{2y-1} t dt,$$

with $x = \frac{2n+1}{2}$ and $y = \frac{1}{2}$. Thus, the beta function become

$$\begin{aligned}
\frac{1}{2} \mathcal{B}(x, y) &= \frac{\Gamma(x)\Gamma(y)}{2\Gamma(x+y)} \\
&= \frac{\sqrt{\pi} \Gamma(\frac{2n+1}{2})}{2 n!} \\
&= \frac{\pi (2n+1)!!}{2 \cdot 2^n \cdot n!}
\end{aligned}$$

The elliptic integral is thereby

$$\begin{aligned}
 K(k) &= \frac{\pi}{2} \sum_{n \geq 0} \left[\frac{(2n-1)!!}{2^n \cdot n!} k^n \right]^2 \\
 &= \frac{\pi}{2} \sum_{n \geq 0} \left[\frac{(2n-1)!!}{(2n)!!} k^n \right]^2.
 \end{aligned}$$

The period of a simple pendulum is

$$T = 2\pi \sqrt{\frac{l}{g}} \sum_{n \geq 0} \left[\frac{(2n-1)!!}{(2n)!!} \sin^n \frac{\phi_0}{2} \right]^2$$

By expanding the sum, we get

$$T = 2\pi \sqrt{\frac{l}{g}} \left(1 + \left(\frac{1}{2} \right)^2 \sin^2 \frac{\phi}{2} + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 \sin^4 \frac{\phi}{2} + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 \sin^6 \frac{\phi}{2} + \dots \right).$$

Using the Maclaurin series

$$\sin \frac{\phi_0}{2} = \frac{1}{2} \phi_0 - \frac{1}{48} \phi_0^3 + \frac{1}{3840} \phi_0^5 - \frac{1}{645120} \phi_0^7 + \dots,$$

we finally get

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$$T = 2\pi \sqrt{\frac{l}{g}} \left(1 + \frac{1}{16} \phi_0^2 + \frac{11}{3072} \phi_0^4 + \dots \right)$$