

MY HUMBLE SOLUTION TO

VOLUME 1 OF COURSE OF THEORETICAL PHYSICS

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**MECHANICS**

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# CHAPTER 1.

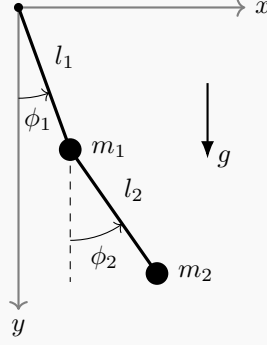
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## THE EQUATIONS OF MOTION

## §5 THE LAGRANGIAN FOR A SYSTEM OF PARTICLES

## PROBLEM 1.

Find the Lagrangian of a coplanar double pendulum when placed in a uniform gravitational field (acceleration  $g$ ).



SOLUTION: The generalized co-ordinates of the system are the two angles  $\phi_1$  and  $\phi_2$ . We need to express the Cartesian co-ordinates in terms of those two angles. First, the Cartesian position of the particle  $m_1$  is

$$\begin{aligned} x_1 &= f_1(\phi_1) = l_1 \sin \phi_1 \\ y_1 &= g_1(\phi_1) = l_1 \cos \phi_1. \end{aligned}$$

By taking the time derivative of those, we obtain

$$\begin{aligned} \dot{x}_1 &= \frac{\partial f_1}{\partial \phi_1} \dot{\phi}_1 = l_1 \cos \phi_1 \dot{\phi}_1 \\ \dot{y}_1 &= \frac{\partial g_1}{\partial \phi_1} \dot{\phi}_1 = -l_1 \sin \phi_1 \dot{\phi}_1. \end{aligned}$$

Substituting these expressions in the Lagrangian, we obtain the desired Lagrangian form, that is

$$\begin{aligned} L_1 &= T_1 - U_1 \\ &= \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + m_1 g y_1 \\ &= \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2 + m_1 g l_1 \cos \phi_1. \end{aligned} \tag{1.1}$$

Second, the Cartesian position of the particle  $m_2$  is

$$\begin{aligned} x_2 &= f_2(\phi_1, \phi_2) = l_1 \sin \phi_1 + l_2 \sin \phi_2 \\ y_2 &= g_2(\phi_1, \phi_2) = l_1 \cos \phi_1 + l_2 \cos \phi_2. \end{aligned}$$

By taking the time derivative of those, we obtain

$$\begin{aligned} \dot{x}_2 &= \sum_k \frac{\partial f_2}{\partial \phi_k} \dot{\phi}_k = l_1 \dot{\phi}_1 \cos \phi_1 + l_2 \dot{\phi}_2 \cos \phi_2 \\ \dot{y}_2 &= \sum_k \frac{\partial g_2}{\partial \phi_k} \dot{\phi}_k = -l_1 \dot{\phi}_1 \sin \phi_1 - l_2 \dot{\phi}_2 \sin \phi_2. \end{aligned}$$

Substituting these expressions in the Lagrangian, we obtain the desired Lagrangian form, that is

$$\begin{aligned} L_2 &= T_2 - U_2 \\ &= \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) + m_2 g y_2 \\ &= \frac{1}{2} m_2 (l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)) + m_2 g (l_1 \cos \phi_1 + l_2 \cos \phi_2), \end{aligned} \tag{1.2}$$

where the angle difference identity as been used. Finally, the Lagrangian of the complete system is simply

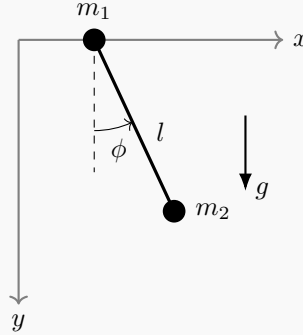
the sum of Eq. (1.1) and Eq. (1.2), thus

ANSWER TO PROBLEM 1.

$$\begin{aligned} L = & \frac{1}{2}(m_1 + m_2)l_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2l_2^2\dot{\phi}_2^2 + m_2l_1l_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2) \\ & + (m_1 + m_2)gl_1\cos\phi_1 + m_2gl_2\cos\phi_2 \end{aligned}$$

## PROBLEM 2.

Find the Lagrangian of a simple pendulum of mass  $m_2$ , with a mass  $m_1$  at the point of support which can move on a horizontal line lying in the plane in which  $m_2$  moves when placed in a uniform gravitational field (acceleration  $g$ ).



SOLUTION: The generalized co-ordinates  $q$  of the system are the position  $x$  and the angle  $\phi$ . Therefore, the Lagrangian of the first particle is simply

$$L_1 = T_1 = \frac{1}{2}m_1\dot{x}^2. \quad (1.3)$$

The Cartesian position of the particle  $m_2$  can be express in terms of the generalized co-ordinates by

$$\begin{aligned} x_2 &= f_2(x, \phi) = x + l \sin \phi \\ y_2 &= g_2(x, \phi) = l \cos \phi. \end{aligned}$$

The time derivative of the position is then

$$\begin{aligned} \dot{x}_2 &= \sum_k \frac{\partial f_2}{\partial q_k} \dot{q}_k = \dot{x} + l\dot{\phi} \cos \phi \\ \dot{y}_2 &= \sum_k \frac{\partial g_2}{\partial q_k} \dot{q}_k = -l\dot{\phi} \sin \phi. \end{aligned}$$

$$\begin{aligned} L_2 &= T_2 - U_2 \\ &= \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) + m_2gy_2 \\ &= \frac{1}{2}m_2(\dot{x}^2 + l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi} \cos \phi) + m_2gl \cos \phi. \end{aligned} \quad (1.4)$$

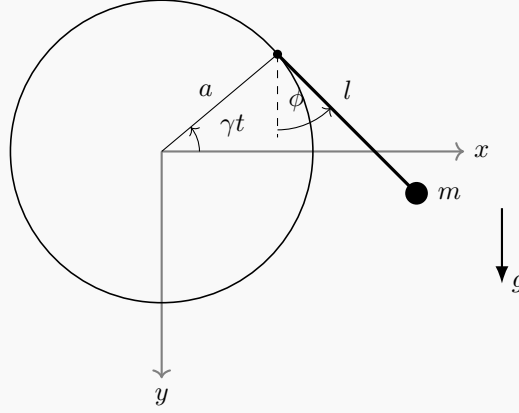
Finally, the Lagrangian of the complete system is the sum of the two Lagrangian Eq. (1.3) and Eq. (1.4), thus

## ANSWER TO PROBLEM 2.

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi} \cos \phi) + m_2gl \cos \phi$$

## PROBLEM 3.

Find the Lagrangian of a simple pendulum of mass  $m$ , when placed in a uniform gravitational field (acceleration  $g$ ), whose point of support ...



(a)

moves uniformly on a vertical circle with constant frequency  $\gamma$ .

SOLUTION: If we set that the rotation of the point of support is counterclockwise, the Cartesian position of the particle  $m$  is

$$\begin{aligned} x &= f(\phi_1) = a \cos \gamma t + l \sin \phi \\ y &= g(\phi_1) = -a \sin \gamma t + l \cos \phi. \end{aligned}$$

The time derivative of the position is then

$$\begin{aligned} \dot{x} &= -a\gamma \sin \gamma t + l\dot{\phi} \cos \phi \\ \dot{y} &= -a\gamma \cos \gamma t - l\dot{\phi} \sin \phi. \end{aligned}$$

The potential energy of the system is

$$\begin{aligned} U &= -mgy \\ &= -mg(-a \sin \gamma t + l \cos \phi). \end{aligned}$$

The term  $mg a \sin \gamma t$  only depend on time and can therefore be ignored (does not contribute to the equations of motion). The kinetic energy of the particle, for its part, is

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2}m \left( l^2 \dot{\phi}^2 + a^2 \gamma^2 + 2a\gamma l \dot{\phi} \sin(\phi - \gamma t) \right), \end{aligned}$$

using again the angle difference identity (I will stop mentioning it). We can observe that the term  $\frac{1}{2}ma^2\gamma^2$  is a constant, thus can be ignored. The last term of the kinetic energy can also be simplified. Indeed,

$$\begin{aligned} ma\gamma l \dot{\phi} \sin(\phi - \gamma t) &= mal\gamma(\dot{\phi} - \gamma) \sin(\phi - \gamma t) + mal\gamma^2 \sin(\gamma t - \phi) \\ &= \frac{d}{dt} (-mal\gamma \cos(\phi - \gamma t)) + mal\gamma^2 \sin(\phi - \gamma t). \end{aligned}$$

After dropping the total time derivative, we can finally get the Lagrangian

ANSWER TO PROBLEM 3 (A).

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2 \sin(\phi - \gamma t) + mgl \cos \phi$$

(b)

oscillates horizontally in the plane of motion of the pendulum according to the law  $x = a \cos \gamma t$ .

SOLUTION: The Cartesian position of the particle  $m$  is

$$\begin{aligned} x &= f(\phi_1) = a \cos \gamma t + l \sin \phi \\ y &= g(\phi_1) = l \cos \phi. \end{aligned}$$

The time derivative of the position is then

$$\begin{aligned} \dot{x} &= -a\gamma \sin \gamma t + l\dot{\phi} \cos \phi \\ \dot{y} &= -l\dot{\phi} \sin \phi. \end{aligned}$$

The potential energy of the system is

$$U = -mgy = -mgl \cos \phi,$$

and the kinetic energy

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2}m \left( l^2 \dot{\phi}^2 + a^2 \gamma^2 \sin^2 \gamma t - 2a\gamma l \dot{\phi} \sin \gamma t \cos \phi \right). \end{aligned}$$

We first see that the term  $\frac{1}{2}ma^2\gamma^2 \sin^2 \gamma t$  only depend on time. The last term of the kinetic energy can also be simplified. Indeed,

$$\begin{aligned} -ma\gamma l \dot{\phi} \sin \gamma t \cos \phi &= -mla\gamma (\gamma \cos \gamma t \sin \phi + \dot{\phi} \sin \gamma t \cos \phi) + mla\gamma^2 \cos \gamma t \sin \phi \\ &= \frac{d}{dt} (-mla\gamma \sin \gamma t \sin \phi) + mla\gamma^2 \cos \gamma t \sin \phi. \end{aligned}$$

After dropping the total time derivative, we can finally get the Lagrangian

ANSWER TO PROBLEM 3 (B).

$$L = \frac{1}{2}ml^2 \dot{\phi}^2 + mla\gamma^2 \cos \gamma t \sin \phi + mgl \cos \phi$$

(c)

oscillates vertically according to the law  $y = a \cos \gamma t$ .

SOLUTION: The Cartesian position of the particle  $m$  is

$$\begin{aligned} x &= f(\phi_1) = l \sin \phi \\ y &= g(\phi_1) = a \cos \gamma t + l \cos \phi. \end{aligned}$$

The time derivative of the position is then

$$\begin{aligned} \dot{x} &= l\dot{\phi} \cos \phi \\ \dot{y} &= -a\gamma \sin \gamma t - l\dot{\phi} \sin \phi. \end{aligned}$$

The potential energy of the system is

$$U = -mgy = -mg(a \cos \gamma t + l \cos \phi),$$

The term  $mga \cos \gamma t$  only depend on time and can therefore be ignored. The kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2}m \left( l^2 \dot{\phi}^2 + a^2 \gamma^2 \sin^2 \gamma t + 2a\gamma l \dot{\phi} \sin \gamma t \sin \phi \right). \end{aligned}$$

We first see that the term  $\frac{1}{2}ma^2\gamma^2 \sin^2 \gamma t$  only depend on time. The last term of the kinetic energy can

also be simplified. Indeed,

$$\begin{aligned} m\gamma l \dot{\phi} \sin \gamma t \sin \phi &= mla\gamma(-\gamma \cos \gamma t \cos \phi + \dot{\phi} \sin \gamma t \sin \phi) + mla\gamma^2 \cos \gamma t \cos \phi \\ &= \frac{d}{dt}(mla\gamma \cos \gamma t \sin \phi) + mla\gamma^2 \cos \gamma t \cos \phi. \end{aligned}$$

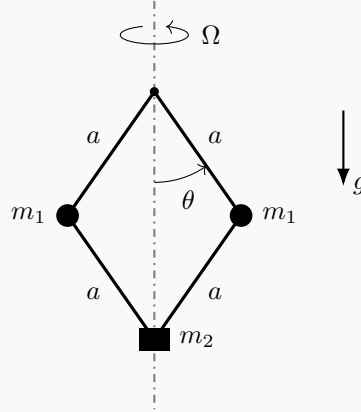
After dropping the total time derivative, we can finally get the Lagrangian

ANSWER TO PROBLEM 3 (c).

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + mla\gamma^2 \cos \gamma t \cos \phi + mgl \cos \phi$$

## PROBLEM 4.

Find the Lagrangian of a simple pendulum of the system below when placed in a uniform gravitational field (acceleration  $g$ ). The particle  $m_2$  moves on a vertical axis and the whole system rotates about this axis with a constant angular velocity  $\Omega$ .



SOLUTION: The position of each particle  $m_1$  is best described in cylindrical coordinates, which is

$$\begin{aligned} r_1 &= a \sin \theta \\ \phi_1 &= \phi \\ z_1 &= a \cos \theta, \end{aligned}$$

where  $\phi$  is the angle of rotation of the system about the axis;  $\dot{\phi} = \Omega$ . The kinetic energy of each particle  $m_1$  is thus

$$\begin{aligned} T_1 &= \frac{1}{2} m_1 (\dot{r}_1^2 + r_1^2 \dot{\phi}_1^2 + \dot{z}_1^2) \\ &= \frac{1}{2} m_1 (a^2 \dot{\theta}^2 \cos^2 \theta + a^2 \Omega^2 \sin^2 \theta + a^2 \dot{\theta}^2 \sin^2 \theta) \\ &= \frac{1}{2} m_1 a^2 (\dot{\theta}^2 + \Omega^2 \sin^2 \theta). \end{aligned} \quad (1.5)$$

The potential energy of this particle can be found by using the  $z$  component of his position, namely

$$\begin{aligned} V_1 &= -m_1 g z_1 \\ &= -m_1 g a \cos \theta. \end{aligned} \quad (1.6)$$

The particle  $m_2$ , in its case, can only move up and down, thus its position can be completely defined by

$$z_2 = 2a \cos \theta.$$

Then, the kinetic energy of each particle  $m_2$  is

$$\begin{aligned} T_2 &= \frac{1}{2} m_2 \dot{z}_2^2 \\ &= m_2 a^2 \dot{\theta}^2 \sin^2 \theta \end{aligned} \quad (1.7)$$

and the potential energy is

$$\begin{aligned} V_2 &= -m_2 g z_2 \\ &= -2m_2 g a \cos \theta. \end{aligned} \quad (1.8)$$

Using Eq. (1.5), Eq. (1.6), Eq. (1.7) and Eq. (1.8), the Lagrangian of the system is

$$L = 2(T_1 + T_2 - V_1 - V_2),$$

therefore

## ANSWER TO PROBLEM 4.

$$L = m_1 a^2 (\dot{\theta}^2 + \Omega^2 \sin^2 \theta) + 2m_2 a^2 \dot{\theta}^2 \sin^2 \theta + 2(m_1 + 2m_2) g a \cos \theta$$

## CHAPTER 2.

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### CONSERVATION LAWS

## §7 MOMENTUM

## PROBLEM 1.

A particle of mass  $m$  moving with velocity  $\mathbf{v}_1$  leaves a half-space in which its potential energy is a constant  $U_1$  and enters another in which its potential energy is a different constant  $U_2$ . Determine the change in the direction of motion of the particle.

SOLUTION: The potential energy only depend on the co-ordinate perpendicular to the plane separating the half-space. Thus, the component of the momentum in that plane is conserved. Denoting by  $\theta_1$  and  $\theta_2$  the angles between the normal to the plane and the velocities  $v_1$  and  $v_2$  of the particle before and after passing the plane, we have

$$\begin{aligned} P_1 \sin \theta_1 &= P_2 \sin \theta_2 \\ \Rightarrow v_1 \sin \theta_1 &= v_2 \sin \theta_2 \\ \Rightarrow \frac{\sin \theta_1}{\sin \theta_2} &= \frac{v_2}{v_1}. \end{aligned} \quad (2.1)$$

The potential energy of the system is also independent of time, therefore the energy of the particle is conserved. Posing  $E_1 = T_1 + U_1$  and  $E_2 = T_2 + U_2$  as the energy of the particle before and after passing the plane, the law of conservation of energy requires

$$\begin{aligned} E_1 &= E_2 \\ \Rightarrow T_1 + U_1 &= T_2 + U_2 \\ \Rightarrow \frac{1}{2}mv_1^2 + U_1 &= \frac{1}{2}mv_2^2 + U_2 \\ \Rightarrow v_2^2 &= v_1^2 + \frac{2}{m}(U_1 - U_2). \end{aligned} \quad (2.2)$$

By substituting Eq. (2.2) in the square of Eq. (2.1), we get

$$\left( \frac{\sin \theta_1}{\sin \theta_2} \right)^2 = \frac{v_1^2 + \frac{2}{m}(U_1 - U_2)}{v_1^2}.$$

After taking the square root, the result is

## ANSWER TO PROBLEM 1.

$$\frac{\sin \theta_1}{\sin \theta_2} = \sqrt{1 + \frac{2}{mv_1^2}(U_1 - U_2)}$$

## §8 CENTRE OF MASS

## PROBLEM 1.

Find the law of transformation of the action  $S$  from one inertial frame to another.

SOLUTION: The Lagrangian  $L$  and  $L'$  of a mechanical system in two inertial frames of reference  $K$  and  $K'$  are respectively

$$L = T - U = \frac{1}{2} \sum_a m_a v_a^2 - U$$

and

$$L' = T' - U = \frac{1}{2} \sum_a m_a v_a'^2 - U.$$

If the frame  $K'$  moves with velocity  $\mathbf{V}$  relative to the frame  $K$ , the velocities of the particles of the mechanical system relative to the two frames are related by  $\mathbf{v}_a = \mathbf{v}_a' + \mathbf{V}$ . We can now express the relation of the Lagrangian of the system in the two frames by

$$\begin{aligned} L &= \frac{1}{2} \sum_a m_a (\mathbf{v}_a' + \mathbf{V})^2 - U \\ &= \frac{1}{2} V^2 \sum_a m_a + \mathbf{V} \cdot \sum_a m_a \mathbf{v}_a' + \frac{1}{2} \sum_a m_a v_a'^2 - U \\ &= L' + \mathbf{V} \cdot \mathbf{P}' + \frac{1}{2} \mu V^2. \end{aligned} \quad (2.3)$$

Integrating Eq. (2.3) with respect to time, we obtain

$$\begin{aligned} S &= \int L dt \\ &= S' + \int (\mathbf{V} \cdot \mathbf{P}') dt + \frac{1}{2} \mu V^2 t \\ &= S' + \mathbf{V} \cdot \int \sum_a m_a \mathbf{v}_a' dt + \frac{1}{2} \mu V^2 t \\ &= S' + \mathbf{V} \cdot \sum_a m_a \mathbf{r}_a' + \frac{1}{2} \mu V^2 t. \end{aligned}$$

The law of transformation of the action  $S$  is then

## ANSWER TO PROBLEM 1.

$$S = S' + \mu \mathbf{V} \cdot \mathbf{R}' + \frac{1}{2} \mu V^2 t$$

## §9 ANGULAR MOMENTUM

## PROBLEM 1.

Obtain the expressions for the Cartesian components and the magnitude of the angular momentum of a particle in cylindrical co-ordinates  $r, \phi, z$ .

SOLUTION: The Cartesian components of the angular momentum are simply

$$\mathbf{M} = \mathbf{r} \times \mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \times m \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = m \begin{bmatrix} y\dot{z} - z\dot{y} \\ z\dot{x} - x\dot{z} \\ x\dot{y} - y\dot{x} \end{bmatrix}. \quad (2.4)$$

In cylindrical co-ordinates, the Cartesian components are expressed by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r \cos \phi \\ r \sin \phi \\ z \end{bmatrix}$$

and

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \dot{r} \cos \phi - r\dot{\phi} \sin \phi \\ \dot{r} \sin \phi + r\dot{\phi} \cos \phi \\ \dot{z} \end{bmatrix}.$$

By substituting those in Eq. (2.4), we get

$$\begin{aligned} \mathbf{M} &= m \begin{bmatrix} y\dot{z} - z\dot{y} \\ z\dot{x} - x\dot{z} \\ x\dot{y} - y\dot{x} \end{bmatrix} \\ &= m \begin{bmatrix} (r\dot{z} - z\dot{r}) \sin \phi - rz\dot{\phi} \cos \phi \\ -(r\dot{z} - z\dot{r}) \cos \phi - rz\dot{\phi} \sin \phi \\ r^2\dot{\phi} \end{bmatrix}. \end{aligned} \quad (2.5)$$

By taking the magnitude of Eq. (2.5), we finally get

## ANSWER TO PROBLEM 1.

$$\begin{aligned} M_x &= m(r\dot{z} - z\dot{r}) \sin \phi - mrz\dot{\phi} \cos \phi \\ M_y &= -m(r\dot{z} - z\dot{r}) \cos \phi - mrz\dot{\phi} \sin \phi \\ M_z &= mr^2\dot{\phi} \\ M^2 &= m^2 r^2 \dot{\phi}^2 (r^2 + z^2) + m^2 (r\dot{z} - z\dot{r})^2 \end{aligned}$$

## PROBLEM 2.

Obtain the expressions for the Cartesian components and the magnitude of the angular momentum of a particle in spherical co-ordinates  $r, \theta, \phi$ .

SOLUTION: In spherical co-ordinates, the Cartesian components are expressed by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r \sin \phi \cos \theta \\ r \sin \phi \sin \theta \\ r \cos \phi \end{bmatrix}$$

and

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \dot{r} \sin \phi \cos \theta + r \dot{\phi} \cos \phi \cos \theta - r \dot{\theta} \sin \phi \sin \theta \\ \dot{r} \sin \phi \sin \theta + r \dot{\phi} \cos \phi \sin \theta + r \dot{\theta} \sin \phi \cos \theta \\ \dot{r} \cos \phi + r \sin \phi \end{bmatrix}.$$

By substituting those in Eq. (2.4) of §9 Prob. 1, we get

$$\begin{aligned} \mathbf{M} &= m \begin{bmatrix} y\dot{z} - z\dot{y} \\ z\dot{x} - x\dot{z} \\ x\dot{y} - y\dot{x} \end{bmatrix} \\ &= m \begin{bmatrix} -r^2(\dot{\theta} \sin \phi + \dot{\phi} \sin \theta \cos \theta \cos \phi) \\ r^2(\dot{\theta} \cos \phi - \dot{\phi} \sin \theta \cos \theta \sin \phi) \\ r^2 \dot{\phi} \sin^2 \theta \end{bmatrix}. \end{aligned} \quad (2.6)$$

By taking the magnitude of Eq. (2.6), we finally get

## ANSWER TO PROBLEM 2.

$$\begin{aligned} M_x &= -mr^2(\dot{\theta} \sin \phi + \dot{\phi} \sin \theta \cos \theta \cos \phi) \\ M_y &= mr^2(\dot{\theta} \cos \phi - \dot{\phi} \sin \theta \cos \theta \sin \phi) \\ M_z &= mr^2 \dot{\phi} \sin^2 \theta \\ M^2 &= m^2 r^4 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) \end{aligned}$$

## PROBLEM 3.

Which components of momentum  $\mathbf{P}$  and angular momentum  $\mathbf{M}$  are conserved in motion in the following fields?

(a)

The field of an infinite homogeneous plane.

SOLUTION: Let us first see how a symmetry of the field leads to the conservation of a component. For a particle of mass  $m$  in a stationary external field  $U$ , the Lagrangian in Cartesian co-ordinates is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z).$$

Since  $P_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$ , Lagrange's equation for  $x$  gives

$$\dot{P}_x = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x}.$$

If the field is unchanged by any translation along  $x$ , *i.e.*  $U(x + a, y, z) = U(x, y, z)$  for every  $a$ , then  $U$  does not depend on  $x$  and  $P_x$  is constant. Any direction can be taken as the  $x$  axis, so a component of  $\mathbf{P}$  is conserved when the field is unchanged by arbitrary translations along that direction.

Similarly, in the cylindrical co-ordinates  $r, \phi, z$ , the Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) - U(r, \phi, z),$$

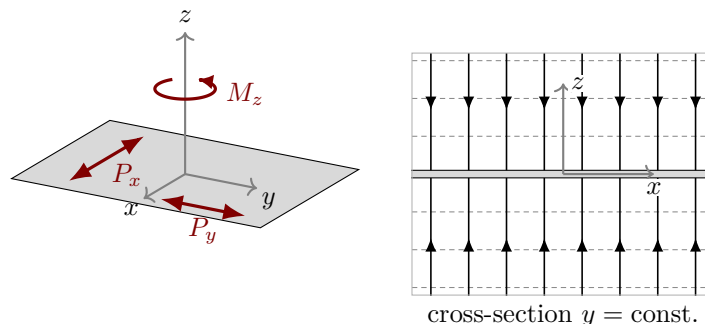
and  $\frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} = M_z$  is the component of the angular momentum along the  $z$  axis found in §9 Prob. 1. Lagrange's equation for  $\phi$  gives

$$\dot{M}_z = \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} = -\frac{\partial U}{\partial \phi}.$$

If the field is unchanged by any rotation about the  $z$  axis, then  $U$  does not depend on  $\phi$  and  $M_z$  is constant. Here, the angular momentum is measured about the symmetry axis, *i.e.* with the origin on that axis. The symmetries involved are continuous: the field must be unchanged by translations or rotations of any amount, however small, for the corresponding derivative of  $U$  to vanish. Invariance under a few isolated rotations is not enough. The same reasoning holds for the total momentum and angular momentum of a system of particles in the field.

In each of the following figures, the body is shown on the left. The red arrows are the transformations leaving its field unchanged (a double arrow for a translation, a circular arrow for a rotation), each labelled with the quantity it conserves. The field is shown on the right in a cross-section. The arrows of the field lines give the direction of the force  $\mathbf{F} = -\nabla U$  on the particle, and the dashed curves are equipotentials, along which  $U$  is constant.

Let the plane be  $z = 0$ , as in the figure. Each point of an infinite homogeneous plane is equivalent to any other, so its field depends only on the distance to the plane, *i.e.*  $U = U(z)$ . It is thus unchanged by any translation along  $x$  or  $y$  and by any rotation about the  $z$  axis: the field lines are perpendicular to the plane and the equipotentials are planes parallel to it.



In terms of the force, this means that

$$F_x = -\frac{\partial U}{\partial x} = 0 \quad \text{and} \quad F_y = -\frac{\partial U}{\partial y} = 0,$$

and the torque about the  $z$  axis vanishes:

$$\dot{M}_z = xF_y - yF_x = 0.$$

However, there is generally a force  $F_z = -dU/dz$  normal to the plane, so  $P_z$  is not conserved, and neither are  $M_x$  and  $M_y$ , whose torques  $yF_z$  and  $-xF_z$  do not vanish. The conserved components are then

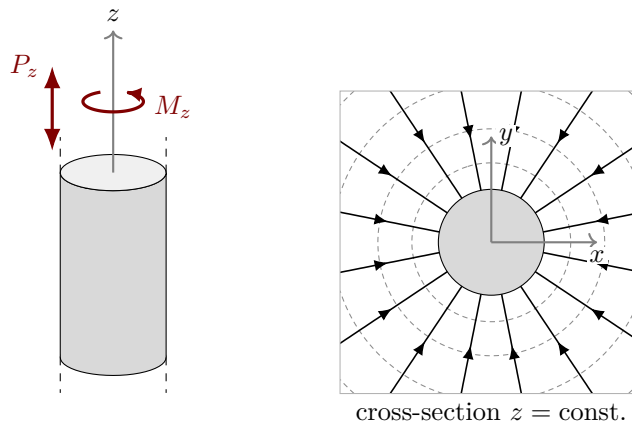
ANSWER TO PROBLEM 3 (A).

$$P_x, \quad P_y, \quad M_z$$

(b)

That of an infinite homogeneous cylinder.

SOLUTION: Let the axis of the cylinder be the  $z$  axis, as in the figure, and  $r$  the distance from that axis. The field of an infinite homogeneous cylinder is the same at every height and in every direction around the axis, so it depends only on  $r$ , *i.e.*  $U = U(r)$ . It is thus unchanged by any translation along  $z$  and by any rotation about the  $z$  axis: in a cross-section, the field lines are radial and the equipotentials are circles.



The force  $\mathbf{F} = -U'(r)\hat{\mathbf{r}}$  is radial. It has no axial component, so  $\dot{P}_z = F_z = 0$ , and its line of action meets the axis, so it produces no torque about it. Indeed,  $(F_x, F_y)$  is parallel to  $(x, y)$  and

$$\dot{M}_z = xF_y - yF_x = 0.$$

The components  $F_x$  and  $F_y$  do not vanish, so  $P_x$  and  $P_y$  are not conserved, nor are  $M_x$  and  $M_y$ . The conserved components are then

ANSWER TO PROBLEM 3 (B).

$$P_z, \quad M_z$$

(c)

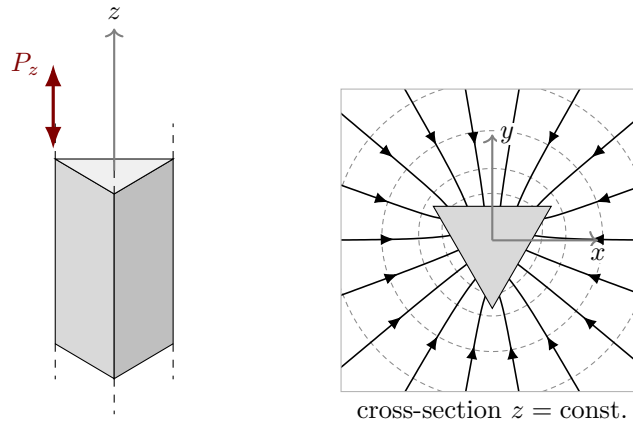
That of an infinite homogeneous prism.

SOLUTION: Let the edges of the prism be parallel to the  $z$  axis, as in the figure. The prism is the same in every cross-section perpendicular to its edges, so its field is unchanged by any translation along

$z$ , *i.e.*  $U = U(x, y)$ , and

$$\dot{P}_z = -\frac{\partial U}{\partial z} = 0.$$

However, its cross-section is a polygon, so the field generally depends on the direction about any axis parallel to the edges. In the figure, the equipotentials near the prism follow the triangular cross-section and only become circular far from it. Even for a regular polygon of  $n$  sides, such as the equilateral triangle shown, the field is unchanged only by the rotations by multiples of  $2\pi/n$  about its axis. These isolated rotations do not make  $\partial U/\partial \phi$  vanish, so they do not imply the conservation of  $M_z$ .



The only conserved component is then

ANSWER TO PROBLEM 3 (c).

$$P_z$$

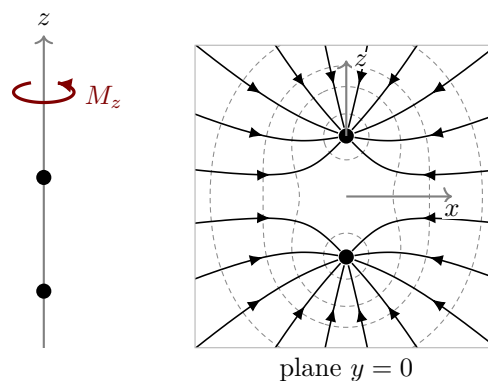
(d)

That of two points.

SOLUTION: Let the two points be on the  $z$  axis, as in the figure. The field is the sum of the fields of the two points, each depending only on the distance from the particle to that point, *i.e.*

$$U = U_1(\rho_1) + U_2(\rho_2),$$

where  $\rho_1$  and  $\rho_2$  are the distances from the particle to the two points. A rotation of the particle about the line joining the points leaves both distances unchanged, so  $U$  does not depend on  $\phi$ . In other words, the force exerted by each point is directed towards that point, so its line of action meets the  $z$  axis and it produces no torque about it. This holds whether or not the two points have the same strength. In the figure, the field lines and equipotentials are shown in a plane containing the axis; they are the same in every such plane.



On the other hand, a translation moves the particle with respect to the fixed points and changes these distances, so no component of  $\mathbf{P}$  is generally conserved. The only conserved component is then

ANSWER TO PROBLEM 3 (D).

$$M_z$$

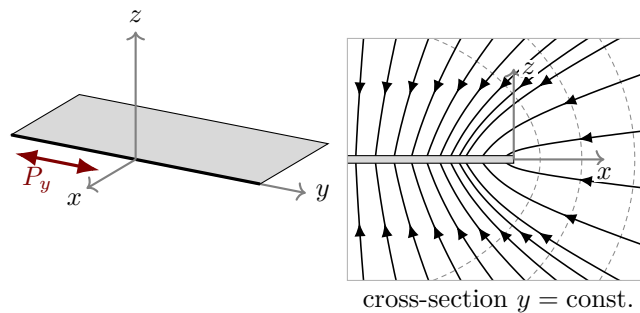
(e)

That of an infinite homogeneous half-plane.

SOLUTION: Let the half-plane be  $z = 0$ ,  $x < 0$ , as in the figure, so that its edge is the  $y$  axis. A translation parallel to the edge maps the half-plane onto itself, so its field is the same in every cross-section  $y = \text{constant}$ , *i.e.*  $U = U(x, z)$ , and

$$\dot{P}_y = -\frac{\partial U}{\partial y} = 0.$$

The edge removes the other symmetries of the whole plane. A translation along  $x$  moves the edge, and the field depends on the distance to it: in the figure, the field lines bend towards the edge and the equipotentials curve around it. A rotation about the  $z$  axis turns the edge, and a rotation about any other axis tilts the half-plane, so no rotation leaves the field unchanged and no component of  $\mathbf{M}$  is generally conserved.



The only conserved component is then

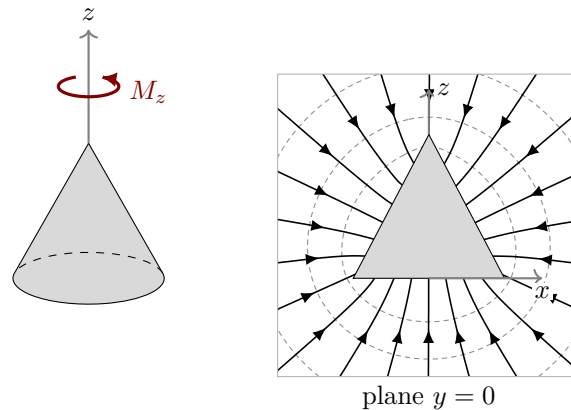
ANSWER TO PROBLEM 3 (E).

$$P_y$$

(f)

That of a homogeneous cone.

SOLUTION: Let the axis of the circular cone be the  $z$  axis, as in the figure. A rotation about its axis maps the cone onto itself, so its field can depend on  $r$  and  $z$  but not on  $\phi$ , *i.e.*  $U = U(r, z)$ , and  $M_z$  is conserved. In the figure, the field lines and equipotentials are shown in a plane containing the axis; they are the same in every such plane.



On the other hand, the apex of the cone is a fixed point. Any translation moves it, even one along the axis, so the translated cone is not the same and the field changes. This would remain true for an infinite cone. Hence, no component of  $\mathbf{P}$  is generally conserved. The only conserved component is then

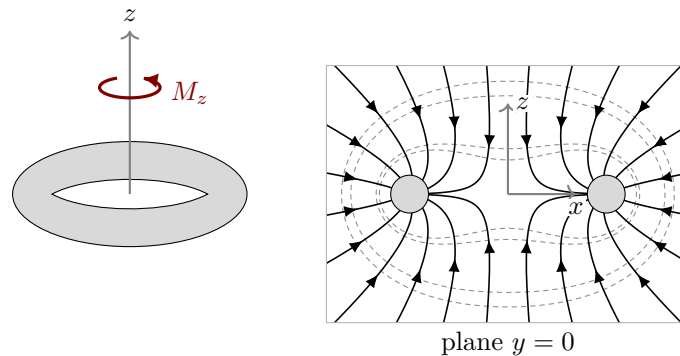
ANSWER TO PROBLEM 3 (F).

$$M_z$$

(g)

That of a homogeneous circular torus.

SOLUTION: Let the axis of the torus be the  $z$  axis, as in the figure. A circular torus is generated by turning a circle about this axis, so any rotation about it maps the torus onto itself. Its field can then depend on  $r$  and  $z$  but not on  $\phi$ , *i.e.*  $U = U(r, z)$ , and  $M_z$  is conserved. In the figure, the field lines and equipotentials are shown in a plane containing the axis; they are the same in every such plane.



On the other hand, the torus has a fixed position, with its centre at the origin and its central circle in the plane  $z = 0$ . Any translation moves it to another position and changes the field, so no component of  $\mathbf{P}$  is generally conserved. The only conserved component is then

ANSWER TO PROBLEM 3 (G).

$$M_z$$

(h)

That of an infinite homogeneous cylindrical helix.

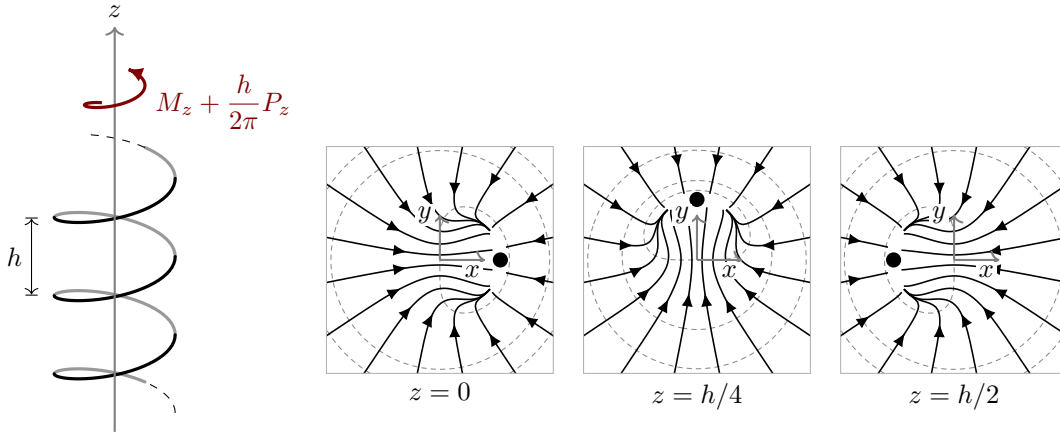
SOLUTION: Let the axis of the helix be the  $z$  axis and  $R$  its radius. As in the figure, its points are

$$(x, y, z) = \left( R \cos \phi, R \sin \phi, \frac{h\phi}{2\pi} \right),$$

where  $h$  is the pitch: the helix advances by  $h$  along the axis during one positive rotation of  $2\pi$ . A rotation by  $\delta\phi$  alone does not map the helix onto itself, nor does an arbitrary translation  $\delta z$  alone (the translations by whole multiples of  $h$  do, but they are isolated transformations, as the rotations of the prism). However, the combined screw displacement

$$\delta z = \frac{h}{2\pi} \delta\phi$$

maps each point of the helix to the point at  $\phi + \delta\phi$ , so it does preserve the helix and its field. Consequently, the field in the cross-section at the height  $z$  is the one at  $z = 0$  turned by the angle  $2\pi z/h$ . This is what the three cross-sections of the figure show: increasing  $z$  by  $h/4$  turns the pattern by  $\pi/2$ . In these cross-sections, the field lines show the components of the field in each plane, and not the complete three-dimensional field lines, since the field also has a component along  $z$ .



In cylindrical co-ordinates, the field thus depends on  $\phi$  and  $z$  only through the combination

$$\xi = z - \frac{h}{2\pi} \phi, \quad U = U(r, \xi).$$

Since  $\phi$  and  $\phi + 2\pi$  describe the same point while  $\xi$  changes by  $h$ , the function  $U(r, \xi)$  is periodic in  $\xi$  with period  $h$ . The Lagrangian is

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2 + \dot{z}^2) - U \left( r, z - \frac{h\phi}{2\pi} \right),$$

and its derivatives with respect to  $\phi$  and  $z$  are

$$\frac{\partial L}{\partial \phi} = \frac{h}{2\pi} \frac{\partial U}{\partial \xi} \quad \text{and} \quad \frac{\partial L}{\partial z} = - \frac{\partial U}{\partial \xi}.$$

Using Lagrange's equations for  $\phi$  and  $z$ , with  $M_z = mr^2 \dot{\phi}$  and  $P_z = m\dot{z}$ , we get

$$\dot{M}_z = \frac{h}{2\pi} \frac{\partial U}{\partial \xi} \quad \text{and} \quad \dot{P}_z = - \frac{\partial U}{\partial \xi},$$

so that

$$\frac{d}{dt} \left( M_z + \frac{h}{2\pi} P_z \right) = \frac{h}{2\pi} \frac{\partial U}{\partial \xi} - \frac{h}{2\pi} \frac{\partial U}{\partial \xi} = 0.$$

No individual Cartesian component of  $\mathbf{P}$  or  $\mathbf{M}$  is generally conserved here. The screw symmetry conserves a combination of the axial angular momentum and the axial momentum, whose two terms have the same dimensions since  $h$  is a length. The conserved quantity is then

ANSWER TO PROBLEM 3 (H).

$$M_z + \frac{h}{2\pi} P_z = \text{constant}$$

## §10 MECHANICAL SIMILARITY

### PROBLEM 1.

Find the ratio of the times in the same path for particles having different masses but the same potential energy.

SOLUTION: If the two particles have the same path, then the ratio of linear dimension is

$$\frac{l'}{l} = \alpha = 1. \quad (2.7)$$

We can define the ratio of time and mass by

$$\frac{t'}{t} = \beta$$

and

$$\frac{m'}{m} = \gamma.$$

Then, the ratio of kinetic energy is

$$\frac{T'}{T} = \frac{m'\mathbf{v}'}{m\mathbf{v}} = \frac{m'}{m} \left( \frac{d\mathbf{r}'}{d\mathbf{r}} \frac{dt}{dt'} \right)^2 = \frac{\gamma\alpha^2}{\beta^2}.$$

To leave the equation of motion unaltered, the ratio of the kinetic energy and the potential energy must be the same, *i.e.* ,

$$\begin{aligned} \frac{U'}{U} &= \frac{T'}{T} \\ \Rightarrow \alpha^k &= \frac{\gamma\alpha^2}{\beta^2}. \end{aligned}$$

Using Eq. (2.7), we get

$$\begin{aligned} 1 &= \frac{\gamma}{\beta^2} \\ \Rightarrow \beta &= \sqrt{\gamma}. \end{aligned}$$

The ratio of the times is then

### ANSWER TO PROBLEM 1.

$$\frac{t'}{t} = \sqrt{\frac{m'}{m}}$$

## PROBLEM 2.

Find the ratio of the times in the same path for particles having the same mass but potential energies differing by a constant factor.

SOLUTION: We proceed as in §10 Prob. 1. If the two particles have the same path, then the ratio of linear dimension is

$$\frac{l'}{l} = \alpha = 1. \quad (2.8)$$

We can define the ratio of time by

$$\frac{t'}{t} = \beta.$$

Then, the ratio of kinetic energy is

$$\frac{T'}{T} = \frac{\alpha^2}{\beta^2}.$$

The potential energy of the two particles differ by a constant factor, which mean that

$$\frac{U'}{U} = \gamma.$$

To leave the equation of motion unaltered, the ratio of the kinetic energy and the potential energy must be the same, *i.e.* ,

$$\begin{aligned} \frac{U'}{U} &= \frac{T'}{T} \\ \Rightarrow \gamma &= \frac{\alpha^2}{\beta^2}. \end{aligned}$$

Using Eq. (2.8), we get

$$\begin{aligned} \gamma &= \frac{1}{\beta^2} \\ \Rightarrow \beta &= \sqrt{\frac{1}{\gamma}}. \end{aligned}$$

The ratio of the times is then

## ANSWER TO PROBLEM 2.

$$\frac{t'}{t} = \sqrt{\frac{U}{U'}}$$

## CHAPTER 3.

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### INTEGRATION OF THE EQUATIONS OF MOTION

## §11 MOTION IN ONE DIMENSION

## PROBLEM 1.

Determine the period of oscillations of a simple pendulum (a particle of mass  $m$  suspended by a string of length  $l$  in a gravitational field) as a function of the amplitude of the oscillations.

SOLUTION: The potential energy of a simple pendulum is

$$U(\phi) = -mgl \cos \phi,$$

see §5 Prob. 1. If we define  $\phi_0$  as the maximum value of  $\phi$ , the potential energy is equal to the total energy at this point, that is

$$U(\phi_0) = E = -mgl \cos \phi_0.$$

The energy of the pendulum could also be write using the kinetic energy,

$$E = \frac{1}{2}ml^2\dot{\phi}^2 + U(\phi).$$

This first order differential equation can be integrate to give the period of oscillation

$$\begin{aligned} T &= 2\sqrt{\frac{ml^2}{2}} \int_{-\phi_0}^{\phi_0} \frac{d\phi}{\sqrt{E - U(\phi)}} \\ &= 4\sqrt{\frac{ml^2}{2}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{mgl \cos \phi - mgl \cos \phi_0}} \\ &= 4\sqrt{\frac{l}{2g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\cos \phi - \cos \phi_0}}. \end{aligned}$$

To solve this integral, we need first to use a trigonometric identity (I'll let you find or derive this identity), giving us

$$T = 2\sqrt{\frac{l}{g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\sin^2 \frac{\phi_0}{2} - \sin^2 \frac{\phi}{2}}}.$$

Next, we use the substitution

$$\sin \xi = \frac{\sin \frac{\phi}{2}}{\sin \frac{\phi_0}{2}}.$$

It's differential form is

$$\begin{aligned} \frac{d\phi}{d\xi} &= \frac{d}{d\xi} \left( 2 \arcsin \left( \sin \frac{\phi_0}{2} \sin \xi \right) \right) \\ &= \frac{2 \sin \frac{\phi_0}{2} \cos \xi}{\sqrt{1 - \sin^2 \frac{\phi_0}{2} \sin^2 \xi}} \end{aligned}$$

The substitution, finally, give

$$\begin{aligned}
T &= 2\sqrt{\frac{l}{g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\sin^2 \frac{\phi_0}{2} - \sin^2 \xi \sin^2 \frac{\phi_0}{2}}} \\
&= 2\sqrt{\frac{l}{g}} \int_0^{\phi_0} \frac{d\phi}{\sin \frac{\phi_0}{2} \sqrt{1 - \sin^2 \xi}} \\
&= 2\sqrt{\frac{l}{g}} \int_0^{\phi_0} \frac{d\phi}{\sin \frac{\phi_0}{2} \cos \xi} \\
&= 2\sqrt{\frac{l}{g}} \int_{\arcsin(0)}^{\arcsin(1)} \frac{2 \sin \frac{\phi_0}{2} \cos \xi d\xi}{\sin \frac{\phi_0}{2} \cos \xi \sqrt{1 - \sin^2 \frac{\phi_0}{2} \sin^2 \xi}} \\
&= 4\sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} \frac{d\xi}{\sqrt{1 - \sin^2 \frac{\phi_0}{2} \sin^2 \xi}}.
\end{aligned}$$

We can also write

$$T = 4\sqrt{\frac{l}{g}} K\left(\sin \frac{\phi_0}{2}\right),$$

where

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\xi}{\sqrt{1 - k^2 \sin^2 \xi}}. \quad (3.1)$$

The integral  $K$  is known as the complete elliptic integral of the first kind. To solve this integral, we first need to see that the integrand is of the form

$$f(x) = (1 - x)^{-\frac{1}{2}}.$$

The  $n$  derivative of this function is

$$f^{(n)}(x) = \frac{(2n-1)!!}{2^n} (1-x)^{-\frac{2n+1}{2}},$$

thus the Maclaurin Series is

$$f(x) = \sum_{n \geq 0} \frac{(2n-1)!!}{2^n \cdot n!} x^n.$$

Using this result in Eq. (3.1), we get

$$\begin{aligned}
K(k) &= \int_0^{\frac{\pi}{2}} \sum_{n \geq 0} \frac{(2n-1)!!}{2^n \cdot n!} k^{2n} \sin^{2n} \xi d\xi \\
&= \sum_{n \geq 0} \frac{(2n-1)!!}{2^n \cdot n!} k^{2n} \int_0^{\frac{\pi}{2}} \sin^{2n} \xi d\xi.
\end{aligned}$$

This last integral can be solve using the beta function

$$\mathcal{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2 \int_0^{\frac{\pi}{2}} \sin^{2x-1} t \cos^{2y-1} t dt,$$

with  $x = \frac{2n+1}{2}$  and  $y = \frac{1}{2}$ . Thus, the beta function become

$$\begin{aligned}
\frac{1}{2} \mathcal{B}(x, y) &= \frac{\Gamma(x)\Gamma(y)}{2\Gamma(x+y)} \\
&= \frac{\sqrt{\pi} \Gamma(\frac{2n+1}{2})}{2 n!} \\
&= \frac{\pi (2n+1)!!}{2 \cdot 2^n \cdot n!}
\end{aligned}$$

The elliptic integral is thereby

$$\begin{aligned}
 K(k) &= \frac{\pi}{2} \sum_{n \geq 0} \left[ \frac{(2n-1)!!}{2^n \cdot n!} k^n \right]^2 \\
 &= \frac{\pi}{2} \sum_{n \geq 0} \left[ \frac{(2n-1)!!}{(2n)!!} k^n \right]^2.
 \end{aligned}$$

The period of a simple pendulum is

$$T = 2\pi \sqrt{\frac{l}{g}} \sum_{n \geq 0} \left[ \frac{(2n-1)!!}{(2n)!!} \sin^n \frac{\phi_0}{2} \right]^2$$

By expanding the sum, we get

$$T = 2\pi \sqrt{\frac{l}{g}} \left( 1 + \left( \frac{1}{2} \right)^2 \sin^2 \frac{\phi}{2} + \left( \frac{1 \cdot 3}{2 \cdot 4} \right)^2 \sin^4 \frac{\phi}{2} + \left( \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 \sin^6 \frac{\phi}{2} + \dots \right).$$

Using the Maclaurin series

$$\sin \frac{\phi_0}{2} = \frac{1}{2} \phi_0 - \frac{1}{48} \phi_0^3 + \frac{1}{3840} \phi_0^5 - \frac{1}{645120} \phi_0^7 + \dots,$$

we finally get

ANSWER TO PROBLEM 1.

$$T = 2\pi \sqrt{\frac{l}{g}} \left( 1 + \frac{1}{16} \phi_0^2 + \frac{11}{3072} \phi_0^4 + \dots \right)$$

## PROBLEM 2.

Determine the period of oscillation, as a function of the energy, when a particle of mass  $m$  moves in fields for which the potential energy is

(a)

$$U = A|x|^n$$

SOLUTION: The total energy of the particle is

$$E = \frac{1}{2}m\dot{x}^2 + U(x).$$

Knowing that the maximum value of  $E$  is at  $x_1$ , this position is

$$\begin{aligned} E &= U(x_1) = A|x_1|^n \\ \Rightarrow |x_1| &= \left(\frac{E}{A}\right)^{1/n}, \\ \Rightarrow x_1 &= \pm \left(\frac{E}{A}\right)^{1/n}, \end{aligned}$$

we get the period of oscillation,

$$\begin{aligned} T &= 2\sqrt{\frac{m}{2}} \int_{-\left(\frac{E}{A}\right)^{1/n}}^{\left(\frac{E}{A}\right)^{1/n}} \frac{dx}{\sqrt{E - A|x|^n}} \\ &= 2\sqrt{2m} \int_0^{\left(\frac{E}{A}\right)^{1/n}} \frac{dx}{\sqrt{E - A|x|^n}}. \end{aligned}$$

It is possible to remove the absolute value, because the integral is over positive  $x$ , thus

$$\begin{aligned} T &= 2\sqrt{2m} \int_0^{\left(\frac{E}{A}\right)^{1/n}} \frac{dx}{\sqrt{E - Ax^n}} \\ &= 2\sqrt{\frac{2m}{E}} \int_0^{\left(\frac{E}{A}\right)^{1/n}} \frac{dx}{\sqrt{1 - \frac{A}{E}x^n}}. \end{aligned}$$

Using the substitution  $y = \left(\frac{A}{E}\right)^{\frac{1}{n}} x$ , we get

$$\begin{aligned} T &= 2\sqrt{\frac{2m}{E}} \int_0^1 \frac{\left(\frac{A}{E}\right)^{\frac{1}{n}} dy}{\sqrt{1 - \frac{A}{E} \left(\left(\frac{A}{E}\right)^{\frac{-1}{n}} y\right)^n}} \\ &= 2\sqrt{\frac{2m}{E}} \left(\frac{E}{A}\right)^{\frac{1}{n}} \int_0^1 \frac{dy}{\sqrt{1 - y^n}}. \end{aligned}$$

Using another substitution  $u = y^n$ , we get

$$T = 2\sqrt{\frac{2m}{E}} \left(\frac{E}{A}\right)^{\frac{1}{n}} \int_0^1 \frac{u^{\frac{1}{n}} du}{nu\sqrt{1 - u}}.$$

It is possible to express this integral in term of the beta function

$$\mathcal{B}(z_1, z_2) = \int_0^1 t^{z_1-1} (1-t)^{z_2-1} dt = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)},$$

with  $z_1 = \frac{1}{n}$  and  $z_2 = \frac{1}{2}$ . This give us

$$T = \frac{2}{n} \sqrt{\frac{2m}{E}} \left( \frac{E}{A} \right)^{\frac{1}{n}} \frac{\Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{n}\right)}.$$

Knowing that  $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ , we finally have

ANSWER TO PROBLEM 2 (A).

$$T = \frac{2}{n} \sqrt{\frac{2\pi m}{E}} \left( \frac{E}{A} \right)^{\frac{1}{n}} \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{n}\right)}$$

(b)

$$U = \frac{-U_0}{\cosh^2 \alpha x}, \quad -U_0 < E < 0$$

SOLUTION: The two boundaries of the energy are at  $E = 0 \Rightarrow x = x_1$  and  $E = -U_0 \Rightarrow x = 0$ . The period of oscillation is thus

$$\begin{aligned} T &= 2\sqrt{2m} \int_0^{x_1} \frac{dx}{\sqrt{E - U}} \\ &= 2\sqrt{2m} \int_0^{x_1} \frac{dx}{\sqrt{E + \frac{U_0}{\cosh^2 \alpha x}}} \\ &= 2\sqrt{2m} \int_0^{x_1} \frac{\cosh^2 \alpha x \, dx}{\sqrt{E \cosh^2 \alpha x + U_0}}. \end{aligned}$$

Using the identity  $\cosh^2 \alpha x = 1 + \sinh^2 \alpha x$ , we get

$$T = 2\sqrt{2m} \int_0^{x_1} \frac{\cosh^2 \alpha x \, dx}{\sqrt{E(1 + \sinh^2 \alpha x) + U_0}}.$$

The substitution  $y = \sinh \alpha x$ , give us  $dy = \alpha \cosh \alpha x dx$  and

$$\begin{aligned} y(x_1) &= \sinh \alpha x \\ &= \sqrt{\sinh^2 \alpha x} \\ &= \sqrt{\cosh^2 \alpha x - 1} \\ &= \sqrt{\frac{-U_0}{E} - 1}. \end{aligned}$$

Using this substitution, we get

$$\begin{aligned} T &= \frac{2}{\alpha} \sqrt{2m} \int_0^{\sqrt{\frac{-U_0}{E} - 1}} \frac{dy}{E(1 + y^2) + U_0} \\ &= \frac{2}{\alpha} \sqrt{2m} \int_0^{\sqrt{\frac{-U_0}{E} - 1}} \frac{dy}{E + Ey^2 - E \cosh^2 \alpha x} \end{aligned}$$

Factoring  $|E|$  out of the square root (because  $E < 0$ ), we have

$$\begin{aligned} T &= \frac{2}{\alpha} \sqrt{\frac{2m}{|E|}} \int_0^{\sqrt{\frac{-U_0}{E} - 1}} \frac{dy}{1 - \cosh^2 \alpha x + y^2} \\ &= \frac{2}{\alpha} \sqrt{\frac{2m}{|E|}} \int_0^{\sqrt{\frac{-U_0}{E} - 1}} \frac{dy}{\sqrt{\frac{-U_0}{E} - 1 + y^2}} \end{aligned}$$

This integral is of the form

$$\int_0^a \frac{dz}{\sqrt{a+z^2}} = \sin^{-1} \left( \frac{z}{a} \right) = \frac{\pi}{2}.$$

Thus, the period of oscillation is

ANSWER TO PROBLEM 2 (B).

$$T = \frac{\pi}{\alpha} \sqrt{\frac{2m}{|E|}}$$

(c)

$$U = U_0 \tan^2 \alpha x$$

SOLUTION: The period of oscillation is

$$\begin{aligned} T &= 2\sqrt{2m} \int_0^{x_1} \frac{dx}{\sqrt{E-U}} \\ &= 2\sqrt{2m} \int_0^{x_1} \frac{dx}{\sqrt{E-U_0 \tan^2 \alpha x}} \\ &= 2\sqrt{2m} \int_0^{x_1} \frac{\cos \alpha x \, dx}{\sqrt{E \cos^2 \alpha x - U_0 \sin^2 \alpha x}} \\ &= 2\sqrt{2m} \int_0^{x_1} \frac{\cos \alpha x \, dx}{\sqrt{E - (E+U_0) \sin^2 \alpha x}}. \end{aligned}$$

Using the substitution  $y = i \sin \alpha x$  ( $dy = i\alpha \cos \alpha x$ ), we get

$$\begin{aligned} T &= \frac{2\sqrt{2m}}{i\alpha} \int_0^{i \sin \alpha x_1} \frac{dy}{\sqrt{E + (E+U_0)y^2}} \\ &= \frac{2}{i\alpha} \sqrt{\frac{2m}{E+U_0}} \int_0^{i \sin \alpha x_1} \frac{dy}{\sqrt{\frac{E}{(E+U_0)} + y^2}} \\ &= \frac{2}{i\alpha} \sqrt{\frac{2m}{E+U_0}} \sin^{-1} \left( \frac{i \sin \alpha x_1}{\sqrt{\frac{E}{(E+U_0)}}} \right) \end{aligned}$$

Knowing

$$x_1 = \frac{1}{\alpha} \tan^{-1} \sqrt{\frac{E}{U_0}},$$

we finally get

$$\begin{aligned}
T &= \frac{2}{i\alpha} \sqrt{\frac{2m}{E+U_0}} \sin^{-1} \left( \frac{i \sin \left( \tan^{-1} \sqrt{\frac{E}{U_0}} \right)}{\sqrt{\frac{E}{(E+U_0)}}} \right) \\
&= \frac{2}{i\alpha} \sqrt{\frac{2m}{E+U_0}} \sin^{-1} \left( \frac{i \sqrt{\frac{E}{U_0}}}{\sqrt{1 + \frac{E}{U_0} \sqrt{\frac{E}{(E+U_0)}}}} \right) \\
&= \frac{2}{i\alpha} \sqrt{\frac{2m}{E+U_0}} \sin^{-1} i \\
&= \frac{2}{i\alpha} \sqrt{\frac{2m}{E+U_0}} \frac{\pi i}{2}.
\end{aligned}$$

Thus, the period of oscillation is

ANSWER TO PROBLEM 2 (c).

$$T = \frac{\pi}{\alpha} \sqrt{\frac{2m}{E+U_0}}$$

## §13 THE REDUCED MASS

## PROBLEM 1.

A system consists of one particle of mass  $M$  and  $n$  particles with equal masses  $m$ . Eliminate the motion of the centre of mass and so reduce the problem to one involving  $n$  particles.

SOLUTION: Let  $\mathbf{R}$  be the radius vector of the particle of mass  $M$ , and  $\mathbf{R}_a$  ( $a = 1, 2, \dots, n$ ) those of the particles of mass  $m$ . We put  $\mathbf{r}_a \equiv \mathbf{R}_a - \mathbf{R}$  and take the origin to be at the centre of mass, namely

$$M\mathbf{R} + m \sum_a \mathbf{R}_a = 0.$$

Thus, we got

$$\mathbf{R} = -\frac{m}{M} \left( \sum_a \mathbf{r}_a + n\mathbf{R} \right) = -\frac{m}{\mu} \sum_a \mathbf{r}_a, \quad (3.2)$$

where  $\mu \equiv M + nm$ . The Lagrangian of the system is

$$L = \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} m \sum_a \dot{\mathbf{R}}_a^2 - U.$$

If we substitute Eq. (3.2) and  $\mathbf{R}_a = \mathbf{R} + \mathbf{r}_a$ , we get

$$\begin{aligned} L &= \frac{1}{2} \frac{Mm^2}{\mu^2} \left( \sum_a \dot{\mathbf{r}}_a \right)^2 + \frac{1}{2} nm \dot{\mathbf{R}}^2 + \frac{1}{2} m \sum_a \dot{\mathbf{r}}_a^2 + m \dot{\mathbf{R}} \sum_a \dot{\mathbf{r}}_a - U \\ &= \frac{1}{2} \frac{Mm^2}{\mu^2} \left( \sum_a \dot{\mathbf{r}}_a \right)^2 + \frac{1}{2} \frac{m^3}{\mu^2} \left( \sum_a \dot{\mathbf{r}}_a \right)^2 + \frac{1}{2} m \sum_a \dot{\mathbf{r}}_a^2 - \frac{m^2}{\mu} \left( \sum_a \dot{\mathbf{r}}_a \right)^2 - U \\ &= \frac{1}{2} m \sum_a \dot{\mathbf{r}}_a^2 - \frac{1}{2} \frac{m^2}{\mu} \left( \frac{2\mu - M - nm}{\mu} \right) \left( \sum_a \dot{\mathbf{r}}_a \right)^2. \end{aligned}$$

The Lagrangian is thus only a function of  $\mathbf{r}_a$  and it's time derivative, which is

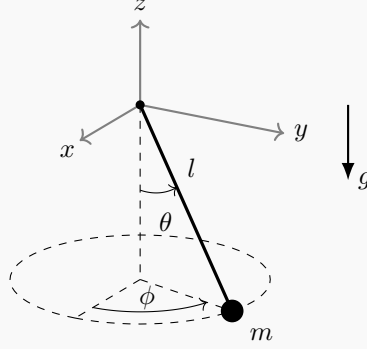
## ANSWER TO PROBLEM 1.

$$L = \frac{1}{2} m \sum_a \dot{\mathbf{r}}_a^2 - \frac{1}{2} \frac{m^2}{\mu} \left( \sum_a \dot{\mathbf{r}}_a \right)^2$$

## §14 MOTION IN A CENTRAL FIELD

## PROBLEM 1.

Integrate the equations of motion for a spherical pendulum (a particle of mass  $m$  moving on the surface of a sphere of radius  $l$  in a gravitational field).



SOLUTION: The Cartesian position of the particle  $m$  is

$$\begin{aligned} x &= l \sin \theta \cos \phi \\ y &= l \sin \theta \sin \phi \\ z &= -l \cos \theta. \end{aligned}$$

The time derivatives of these coordinates are

$$\begin{aligned} \dot{x} &= l\dot{\theta} \cos \theta \cos \phi - l\dot{\phi} \sin \theta \sin \phi \\ \dot{y} &= l\dot{\theta} \cos \theta \sin \phi + l\dot{\phi} \sin \theta \cos \phi \\ \dot{z} &= l\dot{\theta} \sin \theta. \end{aligned}$$

Substituting these expressions in the Lagrangian, we obtain the desired Lagrangian form, that is

$$\begin{aligned} L &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \\ &= \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + mgl \cos \theta. \end{aligned}$$

The Lagrangian does not involve  $\phi$  explicitly, thus this co-ordinate is cyclic and the generalized momentum  $p_\phi$  is conserved. This momentum is the same as the  $z$ -component of angular momentum  $M_z$  (see §9 Prob. 2 with  $r = l$ ) and is written

$$M_z = p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ml^2 \dot{\phi} \sin^2 \theta. \quad (3.3)$$

The energy of the pendulum is

$$E = \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) - mgl \cos \theta.$$

Substituting  $\dot{\phi}$  with Eq. (3.3) in the energy we get

$$\begin{aligned} E &= \frac{1}{2}ml^2\dot{\theta}^2 + \frac{M_z^2}{2ml^2 \sin^2 \theta} - mgl \cos \theta \\ &= \frac{1}{2}ml^2\dot{\theta}^2 + U_{\text{eff}}(\theta), \end{aligned}$$

where  $U_{\text{eff}}(\theta)$  is the effective potential energy. Notice that the energy was re-written as a function of only

one co-ordinate (*i.e.*  $\theta$ ). This is equivalent to the one particle problem we have already see, thus

$$\begin{aligned}\frac{d\theta}{dt} &= \sqrt{\frac{2(E - U_{\text{eff}}(\theta))}{ml^2}} \\ \Rightarrow t &= \int \frac{d\theta}{\sqrt{2(E - U_{\text{eff}}(\theta))}}.\end{aligned}$$

This integral lead to an elliptic integral of the first kind see §11 Prob. 1. We also need to find the solution to the angle  $\phi$ . To do so, we can use the Eq. (3.3) with the chain rule from calculus, that is

$$\begin{aligned}\frac{M_z}{ml^2 \sin^2 \theta} &= \frac{d\phi}{dt} \\ &= \frac{d\phi}{d\theta} \frac{d\theta}{dt} \\ &= \frac{d\phi}{d\theta} \sqrt{\frac{2(E - U_{\text{eff}}(\theta))}{ml^2}}.\end{aligned}$$

Finally, we get

$$\begin{aligned}\frac{d\phi}{d\theta} &= \frac{M_z}{l \sin^2 \theta} \sqrt{\frac{1}{2m(E - U_{\text{eff}}(\theta))}} \\ \Rightarrow \phi &= \frac{M_z}{l} \sqrt{\frac{1}{2m}} \int \frac{d\theta}{\sin^2 \theta (E - U_{\text{eff}}(\theta))}.\end{aligned}$$

This integral lead to an elliptic integral of the third kind (I really don't want to solve this). It is possible to find the range of angle  $\theta$ . At the maximum and minimum value of  $\theta$ , the pendulum as no kinetic energy, thus

$$\begin{aligned}E &= U_{\text{eff}} \\ &= \frac{M_z^2}{2ml^2 \sin^2 \theta} - mgl \cos \theta \\ &= \frac{M_z^2}{2ml^2 (1 - \cos^2 \theta)} - mgl \cos \theta\end{aligned}$$

We can find a cubic algebraic equation for  $\cos \theta$ , that is

$$\begin{aligned}0 &= (E + mgl \cos \theta) (2ml^2 (1 - \cos^2 \theta)) - M_z^2 \\ &= mgl (\cos^3 \theta - \cos \theta) + E (\cos^2 \theta - 1) + \frac{M_z^2}{2ml^2}.\end{aligned}$$

The centrifugal part of the effective potential  $U_{\text{eff}}$ , namely

$$\frac{M_z^2}{2ml^2 \sin^2 \theta},$$

must be positive. As we can see, this term diverge at  $\theta \rightarrow 0$  and  $\theta \rightarrow \pi$ , thus the angle  $\theta$  is bound. This can be write as

$$\theta \in [\theta_p, \theta_a] \qquad 0 < \theta_p \leq \theta_a < \pi,$$

where the subscripts of the angles  $\theta_p$  and  $\theta_a$  are for the perigee and apogee. This mean that the two roots of the cubic algebraic equation for  $\cos \theta$  is bound between  $-1$  and  $+1$ . To summarize, the solution of the equation of motion of the spherical pendulum is

#### ANSWER TO PROBLEM 1.

$$t = \int_{\theta_p}^{\theta_a} \frac{d\theta}{\sqrt{2(E - U_{\text{eff}}(\theta))}} \qquad \phi = \frac{M_z}{l} \sqrt{\frac{1}{2m}} \int_{\theta_p}^{\theta_a} \frac{d\theta}{\sin^2 \theta (E - U_{\text{eff}}(\theta))}$$

## PROBLEM 2.

Integrate the equations of motion for a particle moving on the surface of a cone (of vertical angle  $2\alpha$ ) placed vertically and with vertex downwards in a gravitational field.

SOLUTION: We proceed as in §14 Prob. 1. By using the spherical co-ordinate, we can write the position of the particle as

$$\begin{aligned}x &= r \cos \phi \sin \alpha \\y &= r \sin \phi \sin \alpha \\z &= r \cos \alpha.\end{aligned}$$

By taking the time derivative of those, we obtain

$$\begin{aligned}\dot{x} &= \dot{r} \cos \phi \sin \alpha - r \dot{\phi} \sin \phi \sin \alpha \\ \dot{y} &= \dot{r} \sin \phi \sin \alpha + r \dot{\phi} \cos \phi \sin \alpha \\ \dot{z} &= \dot{r} \cos \alpha.\end{aligned}$$

The Lagrangian is thus

$$\begin{aligned}L &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \\ &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 \sin^2 \alpha) - mgr \cos \alpha.\end{aligned}$$

The co-ordinate  $\phi$  is cyclic, thus

$$M_z = p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2 \dot{\phi} \sin^2 \alpha \quad (3.4)$$

is conserved. This is the  $z$ -component of the angular momentum found in §9 Prob. 1, with the distance  $r \sin \alpha$  from the axis. The energy of the particle is

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 \sin^2 \alpha) + mgr \cos \alpha. \quad (3.5)$$

We can rewrite Eq. (3.4) as

$$\dot{\phi}^2 = \frac{M_z^2}{m^2 r^4 \sin^4 \alpha}$$

and substituting in Eq. (3.5), we get

$$\begin{aligned}E &= \frac{1}{2}m\dot{r}^2 + \frac{M_z^2}{2mr^2 \sin^2 \alpha} + mgr \cos \alpha \\ &= \frac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r).\end{aligned}$$

Hence,

$$t = \sqrt{\frac{m}{2}} \int \frac{dr}{\sqrt{E - U_{\text{eff}}(r)}}.$$

From Eq. (3.4), it is possible to write

$$\begin{aligned}\frac{d\phi}{dt} &= \frac{M_z}{mr^2 \sin^2 \alpha} \\ \Rightarrow \frac{d\phi}{dr} \frac{dr}{dt} &= \frac{M_z}{mr^2 \sin^2 \alpha} \\ \Rightarrow \frac{d\phi}{dr} \sqrt{\frac{2(E - U_{\text{eff}}(r))}{m}} &= \frac{M_z}{mr^2 \sin^2 \alpha} \\ \Rightarrow \phi &= \frac{M_z}{\sqrt{2m} \sin^2 \alpha} \int \frac{dr}{r^2 \sqrt{E - U_{\text{eff}}(r)}}.\end{aligned}$$

It is possible to find the range of  $r$ . At the maximum and minimum value of  $r$ , the particle has no kinetic

energy, thus

$$\begin{aligned} E &= U_{\text{eff}} \\ &= \frac{M_z^2}{2mr^2 \sin^2 \alpha} + mgr \cos \alpha \end{aligned}$$

We can find a cubic algebraic equation for  $r$ , that is

$$\begin{aligned} 0 &= r^2 (E - mgr \cos \alpha) - \frac{M_z^2}{2m \sin^2 \alpha} \\ &= mgr^3 \cos \alpha - Er^2 + \frac{M_z^2}{2m \sin^2 \alpha}. \end{aligned}$$

This equation has two positive roots,  $r_p$  and  $r_a$ , which are the turning points of the motion. To summarize, the solution of the equation of motion for a particle moving on the surface of a cone is

ANSWER TO PROBLEM 2.

$$t = \sqrt{\frac{m}{2}} \int_{r_p}^{r_a} \frac{dr}{\sqrt{E - U_{\text{eff}}(r)}} \quad \phi = \frac{M_z}{\sqrt{2m} \sin^2 \alpha} \int_{r_p}^{r_a} \frac{dr}{r^2 \sqrt{E - U_{\text{eff}}(r)}}$$

## PROBLEM 3.

Integrate the equations of motion for a pendulum of mass  $m_2$ , with a mass  $m_1$  at the point of support which can move on a horizontal line lying in the plane in which  $m_2$  moves (see the figure in §5 Prob. 2).

SOLUTION: From §5 Prob. 2, the Lagrangian of the system is

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi) + m_2gl\cos\phi.$$

The co-ordinate  $x$  is cyclic, thus

$$p_x = \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x} + m_2l\dot{\phi}\cos\phi = \text{constant} \quad (3.6)$$

is conserved. It is always possible to find an inertial frame of reference where  $p_x = 0$ , using this frame we get

$$\begin{aligned} (m_1 + m_2)\dot{x} + m_2l\dot{\phi}\cos\phi &= 0 \\ \Rightarrow \int \left( (m_1 + m_2)\dot{x} + m_2l\dot{\phi}\cos\phi \right) dt &= \text{constant} \\ \Rightarrow (m_1 + m_2)x + m_2l\sin\phi &= (m_1 + m_2)R_x = \text{constant}. \end{aligned} \quad (3.7)$$

This express the fact that the centre of mass **R** of the system does not move horizontally. It is also possible to rewrite Eq. (3.6) as

$$\begin{aligned} (m_1 + m_2)\dot{x} + m_2l\dot{\phi}\cos\phi &= 0 \\ \Rightarrow \dot{x} &= \frac{-m_2l\dot{\phi}\cos\phi}{m_1 + m_2} \end{aligned}$$

Plugging this into the energy of the system we get

$$\begin{aligned} E &= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi) - m_2gl\cos\phi \\ &= \frac{m_2^2l^2\dot{\phi}^2\cos^2\phi}{2(m_1 + m_2)} + \frac{1}{2}m_2l^2\dot{\phi}^2 - \frac{m_2^2l^2\dot{\phi}^2\cos^2\phi}{m_1 + m_2} - m_2gl\cos\phi \\ &= \frac{1}{2}m_2l^2\dot{\phi}^2 \left( 1 - \frac{m_2}{m_1 + m_2}\cos^2\phi \right) - m_2gl\cos\phi. \end{aligned}$$

Hence,

$$\begin{aligned} t &= l\sqrt{\frac{m_2}{2}} \int \sqrt{\frac{1 - \frac{m_2}{m_1+m_2}\cos^2\phi}{E + m_2gl\cos\phi}} d\phi \\ &= l\sqrt{\frac{m_2}{2(m_1 + m_2)}} \int \sqrt{\frac{m_1 + m_2\sin^2\phi}{E + m_2gl\cos\phi}} d\phi. \end{aligned}$$

Using Eq. (3.7), we can express the position of the mass  $m_2$  as

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x + l\sin\phi \\ l\cos\phi \end{bmatrix} = \begin{bmatrix} R_x - l\sin\phi \left( 1 - \frac{m_2}{m_1+m_2} \right) \\ l\cos\phi \end{bmatrix}.$$

The path of the particle of mass  $m_2$  is thus an arc of an ellipse center at  $(R_x, 0)$  with horizontal semi-axis  $lm_1/(m_1 + m_2)$  and vertical semi-axis  $l$ . It is possible to see that when  $m_1 \rightarrow \infty$ , the path return to the simple pendulum. The solution of the equations of motion is thus

## ANSWER TO PROBLEM 3.

$$t = l\sqrt{\frac{m_2}{2}} \int \sqrt{\frac{1 - \frac{m_2}{m_1+m_2}\cos^2\phi}{E + m_2gl\cos\phi}} d\phi \quad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} R_x - l\sin\phi \left( 1 - \frac{m_2}{m_1+m_2} \right) \\ l\cos\phi \end{bmatrix}$$

## §15 KEPLER'S PROBLEM

## PROBLEM 1.

Find the time dependence of the co-ordinates of a particle with energy  $E = 0$  moving in a parabola in a field  $U = -\alpha/r$ .

SOLUTION: From formula (14.6) of the book, we have the integral

$$\begin{aligned} t &= \int \frac{dr}{\sqrt{\frac{2}{m}(E - U(r)) - \frac{M^2}{m^2 r^2}}} \\ &= \int \frac{dr}{\sqrt{\frac{2\alpha}{mr} - \frac{M^2}{m^2 r^2}}} \\ &= \int \frac{r dr}{\sqrt{\frac{2\alpha}{m}r - \frac{M^2}{m^2}}}. \end{aligned}$$

We use the substitution

$$\begin{aligned} \frac{m}{M}\eta &= \sqrt{\frac{2\alpha}{m}r - \frac{M^2}{m^2}} \\ \Rightarrow r &= \frac{M^2}{2m\alpha}(1 + \eta^2) = \frac{1}{2}p(1 + \eta^2), \end{aligned} \quad (3.8)$$

with the differential form

$$dr = \frac{M^2}{m\alpha}\eta d\eta.$$

Hence, the integral become

$$\begin{aligned} t &= \frac{M^3}{2m\alpha^2} \int (1 + \eta^2) d\eta \\ &= \frac{M^3}{2m\alpha^2} \left( \eta + \frac{1}{3}\eta^3 \right) \\ &= \sqrt{\frac{mp^3}{\alpha}} \frac{1}{2} \eta \left( \eta + \frac{1}{3}\eta^3 \right). \end{aligned}$$

It is important to specify that the parameter  $\eta$  varies from  $-\infty$  to  $\infty$ . Using Eq. (3.8) and

$$\cos \phi = \frac{p}{r} - 1,$$

it is possible to find the Cartesian co-ordinates

$$x = r \cos \phi = \frac{1}{2}p(1 - \eta^2)$$

and

$$y = \sqrt{r^2 - x^2} = p\eta.$$

The parametric form of the required dependence are thus

## ANSWER TO PROBLEM 1.

$$\begin{aligned} r &= \frac{1}{2}p(1 + \eta^2) & t &= \sqrt{\frac{mp^3}{\alpha}} \frac{1}{2} \eta \left( \eta + \frac{1}{3}\eta^3 \right) \\ x &= \frac{1}{2}p(1 - \eta^2) & y &= p\eta \end{aligned}$$

## PROBLEM 2.

Integrate the equations of motion for a particle in a central field

$$U = -\frac{\alpha}{r^2} \quad (\alpha > 0).$$

SOLUTION: From formula (14.6) of the book, we have the integral

$$\begin{aligned} t &= \int \frac{dr}{\sqrt{\frac{2}{m}(E - U(r)) - \frac{M^2}{m^2 r^2}}} \\ &= \int \frac{dr}{\sqrt{\frac{2E}{m} r^2 + \frac{2\alpha}{m} - \frac{M^2}{m^2}}} \\ &= \sqrt{\frac{m}{2E}} \int \frac{r dr}{\sqrt{r^2 + \left(\frac{\alpha}{E} - \frac{M^2}{2mE}\right)}}. \end{aligned}$$

We use the substitution

$$u = r^2 + \left(\frac{\alpha}{E} - \frac{M^2}{2mE}\right)$$

with the differential form

$$du = 2r dr.$$

Hence, the integral become

$$\begin{aligned} t &= \frac{1}{2} \sqrt{\frac{m}{2E}} \int \frac{du}{\sqrt{u}} \\ &= \sqrt{\frac{m}{2E}} \sqrt{u} \\ &= \sqrt{\frac{m}{2E}} \sqrt{r^2 + \frac{\alpha}{E} - \frac{M^2}{2mE}} \\ &= \frac{1}{E} \sqrt{\frac{m}{2}} \left( Er^2 + \alpha - \frac{M^2}{2m} \right). \end{aligned}$$

Formula (14.7) of the book gives us the equation of the path

$$\begin{aligned} \phi &= \int \frac{M dr}{r^2 \sqrt{2m(E - U(r)) - \frac{M^2}{r^2}}} \\ &= \int \frac{M dr}{r^2 \sqrt{2mE + \frac{2m\alpha}{r^2} - \frac{M^2}{r^2}}}. \end{aligned}$$

Using the substitution

$$u = \frac{1}{r}$$

with the differential form

$$du = -\frac{1}{r^2} dr,$$

the integral become

$$\begin{aligned} \phi &= - \int \frac{du}{\sqrt{2mE + (2m\alpha - M^2)u^2}} \\ &= -\frac{1}{\sqrt{2mE}} \int \frac{du}{\sqrt{1 + (ku)^2}}, \end{aligned} \tag{3.9}$$

with

$$k = \sqrt{\frac{2m\alpha - M^2}{2mE}}. \tag{3.10}$$

From there, the solution must be divided for the three possible cases : **(a)**  $E > 0$ ,  $M^2 > 2m\alpha$ ; **(b)**  $E > 0$ ,

$M^2 < 2m\alpha$ ; (c)  $E < 0$   $M^2 < 2m\alpha$ . It is also interesting to know that the path is a Cotes's spiral<sup>1</sup>.

(b) : Eq. (3.10) is still

$$k = \sqrt{\frac{2m\alpha - M^2}{2mE}}.$$

By using the substitution

$$ku = \sinh \theta$$

with the differential form

$$kdu = \cosh \theta d\theta,$$

the integral Eq. (3.9) become

$$\begin{aligned} \phi &= -\frac{1}{\sqrt{2mE}} \int \frac{\cosh \theta d\theta}{k\sqrt{1 + \sinh^2 \theta}} \\ &= -\frac{1}{\sqrt{2m\alpha - M^2}} \theta \\ &= -\frac{1}{\sqrt{2m\alpha - M^2}} \sinh^{-1} \frac{k}{r}. \end{aligned}$$

The equation of the path for the case (b) is thus

$$\begin{aligned} \frac{1}{r} &= \frac{1}{k} \sinh \left( \phi \sqrt{2m\alpha - M^2} \right) \\ &= \sqrt{\frac{2mE}{2m\alpha - M^2}} \sinh \left( \phi \sqrt{\frac{2m\alpha}{M^2} - 1} \right). \end{aligned}$$

(c) : Eq. (3.10) become

$$k = \frac{\sqrt{2m\alpha - M^2}}{i\sqrt{2m|E|}}.$$

By using the substitution

$$iku = \cosh \theta$$

with the differential form

$$ikdu = \sinh \theta d\theta,$$

the integral Eq. (3.9) become

$$\begin{aligned} \phi &= -\frac{1}{i\sqrt{2m|E|}} \int \frac{\sinh \theta d\theta}{ik\sqrt{1 - \cosh^2 \theta}} \\ &= \frac{1}{\sqrt{2m\alpha - M^2}} \theta \\ &= \frac{1}{\sqrt{2m\alpha - M^2}} \cosh^{-1} \frac{ik}{r}. \end{aligned}$$

The equation of the path for the case (c) is thus

$$\begin{aligned} \frac{1}{r} &= \frac{1}{ik} \cosh \left( \phi \sqrt{2m\alpha - M^2} \right) \\ &= \sqrt{\frac{2m|E|}{2m\alpha - M^2}} \cosh \left( \phi \sqrt{\frac{2m\alpha}{M^2} - 1} \right). \end{aligned}$$

(a) : Eq. (3.10) become

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<sup>1</sup>Whittaker ET (1937). A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, with an Introduction to the Problem of Three Bodies (4th ed.). New York: Dover Publications. pp. 80–83. ISBN 978-0-521-35883-5.

$$k = \frac{i\sqrt{2m\alpha - M^2}}{\sqrt{2mE}}.$$

By using the substitution

$$ku = i \cosh \theta$$

with the differential form

$$kdu = i \sinh \theta d\theta,$$

the integral Eq. (3.9) become

$$\begin{aligned} \phi &= -\frac{1}{\sqrt{2mE}} \int \frac{i \sinh \theta d\theta}{k \sqrt{1 - \cosh^2 \theta}} \\ &= -\frac{i}{\sqrt{2m\alpha - M^2}} \theta \\ &= -\frac{i}{\sqrt{2m\alpha - M^2}} \cosh^{-1} \frac{k}{ir}. \end{aligned}$$

The equation of the path for the case (a) is thus

$$\begin{aligned} \frac{1}{r} &= \frac{i}{k} \cosh \left( \frac{\phi}{i} \sqrt{2m\alpha - M^2} \right) \\ &= \sqrt{\frac{2mE}{M^2 - 2m\alpha}} \cos \left( \phi \sqrt{1 - \frac{2m\alpha}{M^2}} \right). \end{aligned}$$

To summarize the equations of motion for a particle in a central inverse-cube law force is

#### ANSWER TO PROBLEM 2.

(a) for  $E > 0$  and  $\frac{M^2}{2m} > \alpha$ ,

$$\frac{1}{r} = \sqrt{\frac{2mE}{M^2 - 2m\alpha}} \cos \left( \phi \sqrt{1 - \frac{2m\alpha}{M^2}} \right)$$

(b) for  $E > 0$  and  $\frac{M^2}{2m} < \alpha$ ,

$$\frac{1}{r} = \sqrt{\frac{2mE}{2m\alpha - M^2}} \sinh \left( \phi \sqrt{\frac{2m\alpha}{M^2} - 1} \right)$$

(c) for  $E < 0$  and  $\frac{M^2}{2m} < \alpha$ ,

$$\frac{1}{r} = \sqrt{\frac{2m|E|}{2m\alpha - M^2}} \cosh \left( \phi \sqrt{\frac{2m\alpha}{M^2} - 1} \right)$$

In all three cases

$$t = \frac{1}{E} \sqrt{\frac{m}{2}} \left( Er^2 + \alpha - \frac{M^2}{2m} \right)$$

## PROBLEM 3.

When a small correction  $\delta U(r)$  is added to the potential energy  $U = -\alpha/r$ , the paths of finite motion are no longer closed, and at each revolution the perihelion is displaced through a small angle  $\delta\phi$ . Find  $\delta\phi$  when

(a)

$$\delta U = \frac{\beta}{r^2}$$

SOLUTION: From formula (14.10) of the book, we have

$$\begin{aligned}\Delta\phi &= 2 \int_{r_{\min}}^{r_{\max}} \frac{M dr}{r^2 \sqrt{2m(E - U) - \frac{M^2}{r^2}}} \\ &= -2 \frac{\partial}{\partial M} \int_{r_{\min}}^{r_{\max}} \sqrt{2m(E - U) - \frac{M^2}{r^2}} dr \\ &= -2 \frac{\partial}{\partial M} \int_{r_{\min}}^{r_{\max}} \sqrt{2m \left( E + \frac{\alpha}{r} - \delta U \right) - \frac{M^2}{r^2}} dr.\end{aligned}$$

Expanding the integrand in powers of  $\delta U$  involves using a Taylor series expansion. Let's denote the integrand as  $F$ :

$$F = \sqrt{2m \left( E + \frac{\alpha}{r} + \delta U \right) - \frac{M^2}{r^2}}$$

We want to expand  $F$  around  $\delta U = 0$ . The expansion will look like:

$$F = F_0 + F_1 \delta U + F_2 (\delta U)^2 + \dots$$

Here,  $F_0$  is the value of  $F$  at  $\delta U = 0$ ,  $F_1$  is the first derivative with respect to  $\delta U$  at  $\delta U = 0$ , and so on. Let's find the derivatives:

$$\begin{aligned}F_0 &= \sqrt{2m \left( E + \frac{\alpha}{r} \right) - \frac{M^2}{r^2}} \\ F_1 &= \left. \frac{\partial F}{\partial (\delta U)} \right|_{\delta U=0} = \frac{m}{\sqrt{2m \left( E + \frac{\alpha}{r} \right) - \frac{M^2}{r^2}}} \\ F_2 &= \left. \frac{\partial^2 F}{\partial (\delta U)^2} \right|_{\delta U=0} = -\frac{m^2 \delta U}{\left( 2m \left( E + \frac{\alpha}{r} \right) - \frac{M^2}{r^2} \right)^{3/2}}.\end{aligned}$$

Therefore, the expanded expression in powers of  $\delta U$  is:

$$F = F_0 + \frac{m}{\sqrt{2m \left( E + \frac{\alpha}{r} \right) - \frac{M^2}{r^2}}} \delta U - \frac{m^2}{2 \left( 2m \left( E + \frac{\alpha}{r} \right) - \frac{M^2}{r^2} \right)^{3/2}} \delta U^2 + \dots$$

After plugging the expanded expression of  $F$  in the integral, we see that the zero-order term gives  $2\pi$ . The first-order term gives the required change  $\delta\phi$ :

$$\delta\phi = \frac{\partial}{\partial M} \int_{r_{\min}}^{r_{\max}} \frac{2m\delta U}{\sqrt{2m \left( E + \frac{\alpha}{r} \right) - \frac{M^2}{r^2}}}. \quad (3.11)$$

We can change the integration over  $r$  to one over  $\phi$ , along the path of the unperturbed motion, using the substitution

$$\phi = \cos^{-1} \frac{(M/r) - (m\alpha/M)}{\sqrt{2mE + \frac{m^2\alpha^2}{M^2}}}$$

with the differential form

$$\begin{aligned}\frac{r^2}{M}d\phi &= \frac{-1}{\sqrt{1 - \left(\frac{(M/r) - (m\alpha/M)}{\sqrt{2mE + \frac{m^2\alpha^2}{M^2}}}\right)^2}} \frac{1}{\sqrt{2mE + \frac{m^2\alpha^2}{M^2}}} \\ &= \sqrt{2m \left(E + \frac{\alpha}{r}\right) - \frac{M^2}{r^2}}.\end{aligned}$$

Hence, the integral Eq. (3.11) become

$$\begin{aligned}\delta\phi &= \frac{\partial}{\partial M} \left( \frac{2m}{M} \int_0^\pi r^2 \delta U d\phi \right) \\ &= \frac{\partial}{\partial M} \left( \frac{2m}{M} \int_0^\pi r^2 \frac{\beta}{r^2} d\phi \right) \\ &= \frac{\partial}{\partial M} \left( \frac{2\pi m\beta}{M} \right) \\ &= -\frac{2\pi m\beta}{M^2}.\end{aligned}\tag{3.12}$$

Using formula (15.4) of the book we can finally write

ANSWER TO PROBLEM 3 (A).

$$\delta\phi = -\frac{2\pi\beta}{\alpha p}$$

(b)

$$\delta U = \frac{\gamma}{r^3}$$

SOLUTION: From Eq. (3.12), we have

$$\delta\phi = \frac{\partial}{\partial M} \left( \frac{2m}{M} \int_0^\pi \frac{\gamma}{r} d\phi \right).$$

Using formula (15.5) of the book, the integral become

$$\begin{aligned}\delta\phi &= \frac{\partial}{\partial M} \left( \frac{2m}{M} \int_0^\pi \frac{\gamma}{p} (1 + e \cos \phi) d\phi \right) \\ &= \frac{\partial}{\partial M} \left( \frac{2\pi\gamma m}{Mp} \right) \\ &= -\frac{6\pi\alpha\gamma m^2}{M^4}.\end{aligned}$$

The last expression can also be written as

ANSWER TO PROBLEM 3 (B).

$$\delta\phi = -\frac{6\pi\gamma}{\alpha p^2}$$

## CHAPTER 4.

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### COLLISIONS BETWEEN PARTICLES

## §16 DISINTEGRATION OF PARTICLES

## PROBLEM 1.

Find the relation between the angles  $\theta_1, \theta_2$  (in the  $L$  system) after a disintegration into two particles.

SOLUTION: The angles of the particles, in the  $C$  system, are related by

$$\theta_0 = \theta_{10} = \pi - \theta_{20},$$

where  $\theta_0$  as been defined to simplify the notation. From formula (16.5) of the book, we have

$$\begin{aligned} V + v_{10} \cos \theta_0 &= v_{10} \sin \theta_0 \cot \theta_1 \\ V - v_{20} \cos \theta_0 &= v_{20} \sin \theta_0 \cot \theta_2. \end{aligned}$$

We must eliminate  $\theta_0$  from the two equations above. To do so, we can first solve for  $\cos \theta_0$  and  $\sin \theta_0$  which give us

$$\sin \theta_0 = \frac{V + v_{10} \cos \theta_0}{v_{10} \cot \theta_1} = \frac{V - v_{20} \cos \theta_0}{v_{20} \cot \theta_2} \quad (4.1)$$

$$\begin{aligned} \Rightarrow V v_{20} \cot \theta_2 + v_{10} v_{20} \cos \theta_0 \cot \theta_2 &= V v_{10} \cot \theta_1 - v_{10} v_{20} \cos \theta_0 \cot \theta_1 \\ \Rightarrow \cos \theta_0 &= \frac{V (v_{10} \cot \theta_1 - v_{20} \cot \theta_2)}{v_{10} v_{20} (\cot \theta_1 + \cot \theta_2)}. \end{aligned} \quad (4.2)$$

Using the sum of square of Eq. (4.1) and Eq. (4.2), we get

$$\begin{aligned} \left( \frac{V + v_{10} \cos \theta_0}{v_{10} \cot \theta_1} \right)^2 + \left( \frac{V (v_{10} \cot \theta_1 - v_{20} \cot \theta_2)}{v_{10} v_{20} (\cot \theta_1 + \cot \theta_2)} \right)^2 &= 1 \\ \Rightarrow (v_{10} + v_{20})^2 + (v_{10} \cot \theta_1 - v_{20} \cot \theta_2)^2 &= \frac{v_{10}^2 v_{20}^2}{V^2} (\cot \theta_1 + \cot \theta_2)^2 \\ \Rightarrow v_{10}^2 \csc^2 \theta_1 + v_{20}^2 \csc^2 \theta_2 - 2 v_{10} v_{20} \cot \theta_1 \cot \theta_2 &= \frac{v_{10}^2 v_{20}^2 \sin^2 (\theta_1 + \theta_2)}{V^2 \sin^2 \theta_1 \sin^2 \theta_2} \\ \Rightarrow v_{20}^2 \sin^2 \theta_1 + v_{10}^2 \sin^2 \theta_2 - 2 v_{10} v_{20} \sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2 &= \frac{v_{10}^2 v_{20}^2}{V^2} \sin^2 (\theta_1 + \theta_2). \end{aligned}$$

Using formula (16.2) of the book and the relation  $v_{10}/v_{20} = m_2/m_1$ , we obtain

## ANSWER TO PROBLEM 1.

$$\frac{m_2}{m_1} \sin^2 \theta_2 + \frac{m_1}{m_2} \sin^2 \theta_1 - 2 \sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2 = \frac{2\epsilon}{(m_1 + m_2)V^2} \sin^2 (\theta_1 + \theta_2).$$

**PROBLEM 2.**

Find the angular distribution of the resulting particles in the  $L$  system.

**SOLUTION:** The distribution of the resulting particles with respect to the angle  $\theta_0$  is given by formula (16.7) of the book, *i.e.*

$$\frac{1}{2} \sin \theta_0 d\theta_0 = \frac{1}{2} d(\cos \theta_0).$$

Using formula (16.6) of the book yields

$$\frac{1}{2} d(\cos \theta_0) = -\frac{V}{v_0} \sin \theta \cos \theta d\theta \pm \left( \sin \theta \sqrt{1 - (V^2/v_0^2) \sin^2 \theta} + \cos \theta \frac{(V^2/v_0^2) \sin \theta \cos \theta}{\sqrt{1 - (V^2/v_0^2) \sin^2 \theta}} \right) d\theta.$$

After simplification, the angular distribution become

**ANSWER TO PROBLEM 2.**

$$\frac{1}{2} \sin \theta d\theta \left( 2 \frac{V}{v_0} \cos \theta \pm \frac{1 + (V^2/v_0^2) \cos 2\theta}{\sqrt{1 - (V^2/v_0^2) \sin^2 \theta}} \right)$$

## PROBLEM 3.

Determine the range of possible values of the angle  $\theta$  between the directions of motion of the two resulting particles in the  $L$  system.

SOLUTION: The tangent of the angle  $\theta = \theta_1 + \theta_2$  is

$$\tan \theta = \frac{\tan \theta_1 + \tan \theta_2}{1 - \tan \theta_1 \tan \theta_2}.$$

Using formula (16.5) of the book to get  $\tan \theta_1$  and  $\tan \theta_2$ , and using  $\theta_0 = \theta_{10} = \pi - \theta_{20}$ , the last equation become

$$\tan \theta = \frac{(v_{10} + v_{20})V \sin \theta_0}{V^2 + (v_{10} - v_{20})V \cos \theta_0 - v_{10}v_{20}}$$

or equivalently

$$\cot \theta = \frac{V^2 + (v_{10} - v_{20})V \cos \theta_0 - v_{10}v_{20}}{(v_{10} + v_{20})V \sin \theta_0}.$$

In order to find the extrema, we take the derivative with respect to  $\theta_0$ ,

$$\begin{aligned} \frac{\partial \cot \theta}{\partial \theta_0} = 0 &= \frac{(v_{20} - v_{10})V^2 - V \cos \theta_0 (V^2 - v_{10}v_{20})}{(v_{10} + v_{20})V^2 \sin^2 \theta_0} \\ \Rightarrow \cos \theta_0 &= \frac{(v_{20} - v_{10})V}{V^2 - v_{10}v_{20}}. \end{aligned}$$

We know that  $-1 \leq \cos \theta_0 \leq 1$ , thus  $v_{10} < V < v_{20}$ . Moreover, we could calculate the derivative of the cotangent with respect to  $V$  and find the expression:

$$\frac{\partial \cot \theta}{\partial V} = \frac{V^2 + v_{10}v_{20}}{(v_{10} + v_{20})V^2 \sin \theta_0}$$

which is positive for all values of  $V$ .

This is why  $\cot \theta$  may assume all the values between 0 and  $\pi$  when  $v_{10} < V < v_{20}$ , because  $\frac{\partial \cot \theta}{\partial \theta_0} \neq 0$  and  $\frac{\partial \cot \theta}{\partial V} \neq 0$  for all  $V, \theta_0$  in that interval. It is easy to see that if  $V$  is kept constant and  $\theta_0$  is varied between 0 and  $\pi$ ,  $\cot \theta$  spans the interval  $(-\infty, +\infty)$ ; similarly, if  $\theta_0$  is kept constant and  $V$  is varied between 0 and  $+\infty$ ,  $\cot \theta$  spans the interval  $(-\infty, +\infty)$ .

When  $V \leq v_{10}$ , it is easy to see that  $\cot \theta$  has a maximum value. In fact,

$$\lim_{\theta_0 \rightarrow 0^+} \cot(x) = -\infty, \quad \lim_{\theta_0 \rightarrow \pi^-} \cot(x) = -\infty$$

and the maximum value of the cotangent is reached when:

$$\cos \theta_0 = \frac{(v_{20} - v_{10})V}{V^2 - v_{10}v_{20}}$$

The calculation of the maximum value leads to:

$$\cot \theta = \frac{V^2 + (v_{10} - v_{20})V \cos \theta_0 - v_{10}v_{20}}{(v_{10} + v_{20})V \sin \theta_0} = -\frac{\sqrt{(V^2 - v_{10}v_{20})^2 - (v_{20} - v_{10})^2 V^2}}{(v_{10} + v_{20})V}$$

Then, after transforming the cotangent into a sine, one gets:

$$\sin \theta = \frac{(v_{20} + v_{10})V}{V^2 + v_{10}v_{20}}$$

If we put:

$$\alpha = \sin^{-1} \left( \frac{(v_{20} + v_{10})V}{V^2 + v_{10}v_{20}} \right)$$

then, in the interval where  $V \leq v_{10}$ , we have the following ranges for  $\theta$ :  $\pi - \alpha \leq \theta \leq \pi$ . Similarly, when

$V \geq v_{20}$ , it is easy to see that  $\cot \theta$  has a minimum value. In fact,

$$\lim_{\theta_0 \rightarrow 0^+} \cot(x) = +\infty, \quad \lim_{\theta_0 \rightarrow \pi^-} \cot(x) = +\infty$$

and the minimum value of the cotangent is reached when  $\theta = \alpha$ . In this case, the ranges for  $\theta$  are  $0 \leq \theta \leq \alpha$  (the equality  $\theta = 0$  is satisfied only when  $V \rightarrow +\infty$ ). The final solution is thus

ANSWER TO PROBLEM 3.

$$\begin{aligned} \text{if } v_{10} < V < v_{20} &\Rightarrow 0 < \theta < \pi \\ \text{if } V \leq v_{10} &\Rightarrow \pi - \alpha \leq \theta \leq \pi \\ \text{if } V \geq v_{20} &\Rightarrow 0 \leq \theta \leq \alpha \end{aligned}$$

where

$$\alpha = \sin^{-1} \left( \frac{(v_{20} + v_{10})V}{V^2 + v_{10}v_{20}} \right).$$

## §17 ELASTIC COLLISIONS

## PROBLEM 1.

Express the velocity of each particle after a collision between a moving particle ( $m_1$ ) and another at rest ( $m_2$ ) in terms of their directions of motion in the  $L$  system.

SOLUTION: To solve this problem, we need to use Fig. 16 of the book. First, using the law of sines, the magnitude of the momentum of the second particle after the collision is

$$p'_2 = OB \frac{\sin \chi}{\sin \theta_2} = mv \frac{\sin \chi}{\cos(\chi/2)} = 2mv \sin \frac{\chi}{2}.$$

We get formula (17.5) of the book if we isolate the velocity

$$v'_2 = \frac{2m_1 v}{m_1 + m_2} \sin \frac{\chi}{2}.$$

Second, using the law of cosines, we get

$$\begin{aligned} OC^2 &= AO^2 + p_1'^2 - 2AO p_1' \cos \theta_1 \\ \Rightarrow p_1' &= AO \cos \theta_1 \pm \sqrt{AO^2 \cos^2 \theta_1 + OC^2 - AO^2} \\ \Rightarrow v_1' &= \frac{m_1 \cos \theta_1 \pm \sqrt{m_2^2 - m_1^2 \sin^2 \theta_1}}{m_1 + m_2} v_1. \end{aligned}$$

For the case where  $m_2 > m_1$  we also get formula (17.5) of the book, but when  $m_1 > m_2$ , the radical may have either sign. To summarize, the answer is

## ANSWER TO PROBLEM 1.

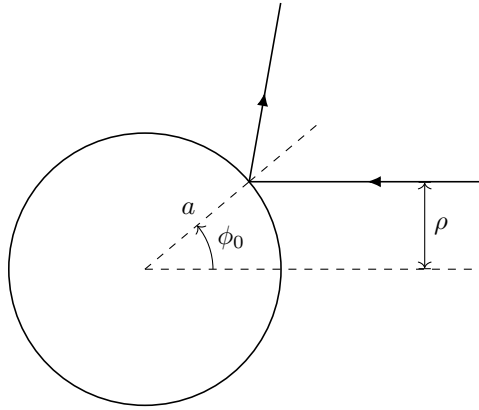
$$\begin{aligned} v'_2 &= \frac{2m_1 v}{m_1 + m_2} \sin \frac{\chi}{2} \\ v'_1 &= \frac{m_1 \cos \theta_1 \pm \sqrt{m_2^2 - m_1^2 \sin^2 \theta_1}}{m_1 + m_2} v_1. \end{aligned}$$

## §18 SCATTERING

## PROBLEM 1.

Determine the effective cross-section for scattering of particles from a perfectly rigid sphere of radius  $a$  (*i.e.* when the interaction is such that  $U = \infty$  for  $r < a$  and  $U = 0$  for  $r > a$ ).

SOLUTION: The particle is reflected from the sphere at the same angle  $\phi_0$  to the normal (*i.e.* the radius) as it is incident, as shown in the figure.



From the figure, we can see

$$\rho = a \sin \phi_0 = a \sin \frac{1}{2}(\pi - \chi) = a \cos \frac{1}{2}\chi. \quad (4.3)$$

Substituting Eq. (4.3) in formula (18.7) or (18.8) of the book, we have

$$d\sigma = \frac{1}{2}\pi a^2 \sin \chi d\chi = \frac{1}{4}a^2 d\chi. \quad (4.4)$$

If we integrate  $d\sigma$  over all angles, we find

$$\sigma = \frac{1}{2}\pi a^2 \int_{-\pi/2}^{\pi/2} \sin \chi d\chi = \pi a^2,$$

which is the projection of the sphere perpendicular to the beam (*i.e.* the impact area). We need to express Eq. (4.4) in the  $L$  system using formula (17.4) of the book. If this equation is solved for  $\cos \theta_1$ , we get, as in §16 Prob. 2 with  $V/v_0$  replaced by  $m_1/m_2$ ,

$$\begin{aligned}
\tan \theta_1 &= \frac{m_2 \sin \chi}{m_1 + m_2 \cos \chi} \\
\Rightarrow \tan^2 \theta_1 \left( \frac{m_1}{m_2} + \cos \chi \right)^2 &= 1 - \cos^2 \chi \\
\Rightarrow \cos \chi &= -\frac{m_1}{m_2} \sin^2 \theta_1 \pm \cos \theta_1 \sqrt{1 - \frac{m_1^2}{m_2^2} \sin^2 \theta_1} \\
\Rightarrow \frac{1}{2} \sin \chi d\chi &= \frac{1}{2} \sin \theta_1 d\theta_1 \left( 2 \frac{m_1}{m_2} \cos \theta_1 \pm \frac{1 + (m_1^2/m_2^2) \cos 2\theta_1}{\sqrt{1 - (m_1^2/m_2^2) \sin^2 \theta_1}} \right).
\end{aligned}$$

We thus get

ANSWER TO PROBLEM 1.

$$\begin{aligned}
d\sigma_1 &= \frac{1}{4} a^2 \left( 2 \frac{m_1}{m_2} \cos \theta_1 \pm \frac{1 + (m_1^2/m_2^2) \cos 2\theta_1}{\sqrt{1 - (m_1^2/m_2^2) \sin^2 \theta_1}} \right) d\theta_1 \\
d\sigma_2 &= a^2 |\cos \theta_2| d\theta_2.
\end{aligned}$$

## PROBLEM 2.

Express the effective cross-section (§18 Prob. 1) as a function of the energy  $\epsilon$  lost by a scattered particle.

SOLUTION: The energy loss by a particle of mass  $m_1$  is equal to the energy gained by the sphere of mass  $m_2$ . Knowing that the sphere is initially at rest, and using  $v'_2$  from §17 Prob. 1, the energy gained is

$$\epsilon = E'_2 = \frac{1}{2}m_2v'^2_2 = \frac{2m_1^2m_2}{(m_1 + m_2)^2}v_\infty^2 \sin^2 \frac{1}{2}\chi = \epsilon_{max} \sin^2 \frac{1}{2}\chi.$$

Using the double angle identity, the variation of  $\epsilon$  is

$$d\epsilon = \epsilon_{max} \sin \frac{1}{2}\chi \cos \frac{1}{2}\chi d\chi = \frac{1}{2}\epsilon_{max} \sin \chi d\chi.$$

Using Eq. (4.4) from §18 Prob. 1, the effective cross section can thus be express as

## ANSWER TO PROBLEM 2.

$$d\sigma = \frac{\pi a^2 d\epsilon}{\epsilon_{max}}$$

**PROBLEM 3.**

Find the effective cross-section as a function of the velocity  $v_\infty$  for particles scattered in a field  $U \sim r^{-n}$ .

SOLUTION: For a central potential field, the scattering angle is determined by the integral

$$\chi = \pi - 2 \int_{r_{min}}^{\infty} \frac{\rho dr}{r^2 \sqrt{1 - \frac{\rho^2}{r^2} - \frac{2U(r)}{mv_\infty^2}}}.$$

Substituting  $U(r) = \alpha r^{-n}$  and define the dimensionless variable  $x = \frac{r}{\rho}$ , then

$$\chi = \pi - 2 \int_{x_{min}}^{\infty} \frac{dx}{x^2 \sqrt{1 - \frac{1}{x^2} - \frac{2\alpha}{mv_\infty^2 \rho^n x^n}}}.$$

Notice the dimensionless combination  $\xi = \frac{2\alpha}{mv_\infty^2 \rho^n}$ , the integral simplifies to

$$\chi = \pi - 2 \int_{x_{min}}^{\infty} \frac{dx}{x^2 \sqrt{1 - \frac{1}{x^2} - \frac{\xi}{x^n}}}.$$

Thus, the scattering angle depends only on  $\xi$ ,

$$\chi = f(\xi),$$

where  $f(\xi)$  is a dimensionless function determined by the integral. To invert this relationship, solve for  $\rho$  in terms of  $\chi$  and  $v_\infty$ ,

$$\xi = \frac{2\alpha}{mv_\infty^2 \rho^n} \Rightarrow \rho = \left( \frac{2\alpha}{mv_\infty^2 \xi} \right)^{1/n}.$$

Let  $\xi = g(\chi)$ , where  $g(\chi)$  is a function encoding the angular dependence. Then,

$$\begin{aligned} \rho &= \left( \frac{2\alpha}{mv_\infty^2} \right)^{1/n} (g(\chi))^{-1/n} \\ \Rightarrow \rho &\sim v_\infty^{-2/n} (g(\chi))^{-1/n}. \end{aligned}$$

The differential cross-section is thus

$$\begin{aligned} d\sigma &= \frac{\rho}{\sin \chi} \left| \frac{d\rho}{d\chi} \right| d\chi \\ d\sigma &\sim v_\infty^{-4/n} \frac{F(\chi)}{\sin \chi} d\chi, \end{aligned}$$

where  $F(\chi) = \left| \frac{d}{d\chi} (g(\chi))^{-1/n} \right|$ . Integrate over the solid angle

$$\sigma \sim v_\infty^{-4/n} \int F(\chi) d\chi.$$

Knowing that the integral depends only on  $n$ , not  $v_\infty$ , we have

**ANSWER TO PROBLEM 3.**

$$\sigma \sim v_\infty^{-4/n}$$

## PROBLEM 4.

Determine the effective cross-section for a particle to “fall” to the centre of a field  $U = -\alpha/r^2$ .

SOLUTION: Let the particle come from infinity with velocity  $v_\infty$  and impact parameter  $\rho$ , and take  $\alpha > 0$ . Its energy and angular momentum are

$$E = \frac{1}{2}mv_\infty^2, \quad M = mv_\infty\rho.$$

The radial motion is governed by the energy equation

$$\frac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r) = E,$$

where the effective potential energy is

$$U_{\text{eff}}(r) = \frac{M^2}{2mr^2} - \frac{\alpha}{r^2} = \frac{1}{r^2} \left( \frac{M^2}{2m} - \alpha \right).$$

This is the field of §15 Prob. 2, where the same condition on  $M^2$  separates the possible paths. Since both terms vary as  $r^{-2}$ , the sign of the bracket determines the behaviour of the particle near the centre. If  $M^2 > 2m\alpha$ , then  $U_{\text{eff}}$  is a repulsive centrifugal barrier which tends to  $+\infty$  as  $r \rightarrow 0$ , so the particle reaches a turning point where  $U_{\text{eff}}(r_{\min}) = E$  and goes back to infinity. If  $M^2 < 2m\alpha$ , then  $U_{\text{eff}} < 0$  for all  $r$ , so  $E - U_{\text{eff}} > 0$  everywhere. The radial velocity never vanishes and an incoming particle has no radial turning point: it falls to the centre. In the limiting case  $M^2 = 2m\alpha$ , we have  $U_{\text{eff}} = 0$ , thus  $\dot{r} = -v_\infty$  and the particle also reaches the centre. The critical trajectory falls to the centre because the centrifugal and attractive terms have the same dependence  $r^{-2}$ . At the critical value of  $M$  they cancel identically, so there is no barrier at all and nothing can stop the particle at a finite distance. This differs from §18 Prob. 5, where the two terms have different powers of  $r$  and  $U_{\text{eff}}$  has a maximum at a finite radius.

The particle therefore falls to the centre if  $M^2 \leq 2m\alpha$ , i.e. if

$$m^2v_\infty^2\rho^2 \leq 2m\alpha \Rightarrow \rho \leq \rho_{\max}, \quad \rho_{\max}^2 = \frac{2\alpha}{mv_\infty^2}.$$

The effective cross-section is obtained by summing the areas of all the annuli of the incident beam that lead to a fall,

$$\sigma = \int_0^{\rho_{\max}} 2\pi\rho \, d\rho = \pi\rho_{\max}^2.$$

We thus get

## ANSWER TO PROBLEM 4.

$$\sigma = \frac{2\pi\alpha}{mv_\infty^2} = \frac{\pi\alpha}{E}$$

## PROBLEM 5.

The same as §18 Prob. 4, but for a field  $U = -\alpha/r^n$  ( $n > 2, \alpha > 0$ ).

SOLUTION: As in §18 Prob. 4, the particle comes from infinity with velocity  $v_\infty$  and impact parameter  $\rho$ , so that

$$E = \frac{1}{2}mv_\infty^2, \quad M = mv_\infty\rho.$$

The effective potential energy is now

$$U_{\text{eff}}(r) = \frac{M^2}{2mr^2} - \frac{\alpha}{r^n}.$$

Since  $n > 2$ , the centrifugal term dominates at large  $r$  and the attractive term dominates at small  $r$ . Thus  $U_{\text{eff}} \rightarrow 0^+$  as  $r \rightarrow \infty$ ,  $U_{\text{eff}} \rightarrow -\infty$  as  $r \rightarrow 0$ , and  $U_{\text{eff}}$  has a maximum at some  $r_b$ , given by

$$U'_{\text{eff}}(r_b) = -\frac{M^2}{mr_b^3} + \frac{n\alpha}{r_b^{n+1}} = 0 \Rightarrow \frac{M^2}{m} = \frac{n\alpha}{r_b^{n-2}}.$$

This barrier analysis assumes  $M \neq 0$ . A head-on particle ( $\rho = 0$ ) has no centrifugal barrier:  $U_{\text{eff}} = -\alpha/r^n$  is monotonic, and the particle falls to the centre. The height of this barrier is

$$U_{\text{eff}}(r_b) = \frac{n\alpha}{2r_b^n} - \frac{\alpha}{r_b^n} = \frac{(n-2)\alpha}{2r_b^n}.$$

If  $E$  is smaller than the barrier height, the particle reaches a turning point outside the barrier and goes back to infinity. If  $E$  exceeds the barrier height, then  $E - U_{\text{eff}} > 0$  for all  $r$ , so the particle has no radial turning point and falls to the centre. Since the barrier height increases with  $M$ , the capture occurs for  $\rho < \rho_{\text{max}}$ , where  $\rho_{\text{max}}$  is the impact parameter for which  $E$  is equal to the barrier height. In this limiting case, the incoming particle approaches asymptotically the unstable circular orbit  $r = r_b$ . Indeed, at  $r = r_b$  both  $E - U_{\text{eff}}$  and  $U'_{\text{eff}}$  vanish, so  $r_b$  is a point of (unstable) equilibrium of the radial motion. Near it,  $\frac{1}{2}m\dot{r}^2 = E - U_{\text{eff}} \propto (r - r_b)^2$ , so the time  $\int dr/|\dot{r}| \propto \int dr/|r - r_b|$  needed to reach  $r_b$  diverges logarithmically. In contrast with §18 Prob. 4, the critical particle therefore never reaches the centre. This single value of  $\rho$  does not affect the cross-section.

Denote by  $r_c$  the position of the maximum at the critical impact parameter. The condition  $E = U_{\text{eff}}(r_c)$  gives

$$\frac{1}{2}mv_\infty^2 = \frac{(n-2)\alpha}{2r_c^n} \Rightarrow r_c^n = \frac{(n-2)\alpha}{mv_\infty^2}.$$

Using  $M = mv_\infty\rho_{\text{max}}$  in the condition for the maximum, we find

$$\begin{aligned} \rho_{\text{max}}^2 &= \frac{M^2}{m^2v_\infty^2} = \frac{n\alpha}{mv_\infty^2r_c^{n-2}} \\ &= \frac{n}{n-2} \frac{r_c^n}{r_c^{n-2}} = \frac{n}{n-2} r_c^2, \end{aligned}$$

where we used  $\alpha/(mv_\infty^2) = r_c^n/(n-2)$  in the second line. The effective cross-section is obtained by summing the areas of all the annuli of the incident beam that lead to a fall,

$$\sigma = \int_0^{\rho_{\text{max}}} 2\pi\rho d\rho = \pi\rho_{\text{max}}^2 = \frac{\pi n}{n-2} r_c^2.$$

We thus get

## ANSWER TO PROBLEM 5.

$$\sigma = \frac{\pi n}{n-2} \left( \frac{(n-2)\alpha}{mv_\infty^2} \right)^{2/n} = \frac{\pi n}{n-2} \left( \frac{(n-2)\alpha}{2E} \right)^{2/n}$$

## PROBLEM 6.

Determine the effective cross-section for particles of mass  $m_1$  to strike a sphere of mass  $m_2$  and radius  $R$  to which they are attracted in accordance with Newton's law.

SOLUTION: We consider the relative motion of the particle and the sphere, with  $r$  the distance between their centres. This is equivalent to the motion of a single particle of reduced mass

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

in the field

$$U(r) = -\frac{\gamma m_1 m_2}{r},$$

where  $\gamma$  is the gravitational constant. Let  $v_\infty$  be the relative velocity at infinity and  $\rho$  the impact parameter. The energy and angular momentum of the relative motion are

$$E = \frac{1}{2} m v_\infty^2, \quad M = m v_\infty \rho,$$

and the radial motion is governed by the energy equation

$$\frac{1}{2} m \dot{r}^2 + \frac{M^2}{2mr^2} - \frac{\gamma m_1 m_2}{r} = E.$$

The distance of closest approach decreases as  $\rho$  decreases, so the particle strikes the sphere if  $\rho \leq \rho_{max}$ , where  $\rho_{max}$  corresponds to the limiting trajectory which grazes the sphere. For this trajectory,  $\dot{r} = 0$  at  $r = R$ . Substituting  $M = m v_\infty \rho_{max}$ , we have

$$\frac{m v_\infty^2 \rho_{max}^2}{2R^2} - \frac{\gamma m_1 m_2}{R} = \frac{1}{2} m v_\infty^2.$$

Multiplying by  $2R^2/(m v_\infty^2)$  and using  $m_1 m_2/m = m_1 + m_2$ ,

$$\begin{aligned} \rho_{max}^2 &= R^2 \left( 1 + \frac{2\gamma m_1 m_2}{m R v_\infty^2} \right) \\ &= R^2 \left( 1 + \frac{2\gamma(m_1 + m_2)}{R v_\infty^2} \right). \end{aligned}$$

The effective cross-section is obtained by summing the areas of all the annuli of the incident beam that lead to a collision with the sphere,

$$\sigma = \int_0^{\rho_{max}} 2\pi \rho \, d\rho = \pi \rho_{max}^2 = \pi R^2 \left( 1 + \frac{2\gamma(m_1 + m_2)}{R v_\infty^2} \right).$$

The second term describes the gravitational focusing of the trajectories towards the sphere. In the high-speed limit  $v_\infty^2 \gg \gamma(m_1 + m_2)/R$ , it becomes negligible and we recover the geometric cross-section  $\sigma = \pi R^2$ . As in the book, we assume  $m_2 \gg m_1$ , so that  $m_1 + m_2 \simeq m_2$ . We thus get

## ANSWER TO PROBLEM 6.

$$\sigma \simeq \pi R^2 \left( 1 + \frac{2\gamma m_2}{R v_\infty^2} \right)$$

## PROBLEM 7.

Deduce the form of a scattering field  $U(r)$ , given the effective cross-section as a function of the angle of scattering for a given energy  $E$ . It is assumed that  $U(r)$  decreases monotonically with  $r$  (a repulsive field), with  $U(0) > E$  and  $U(\infty) = 0$  (O. B. Firsov 1953).

SOLUTION: Let  $d\sigma/d\theta$  be the effective cross-section per unit scattering angle, *i.e.* already integrated over the azimuth. In terms of the cross-section per unit solid angle, since  $d\omega = 2\pi \sin \theta d\theta$ ,

$$\frac{d\sigma}{d\theta} = 2\pi \sin \theta \frac{d\sigma}{d\omega}.$$

As in the book, we assume that the relation between  $\rho$  and  $\chi$  is one-to-one, with  $\chi$  decreasing monotonically from  $\pi$  (head-on collision,  $\rho = 0$ ) to 0 (no deflection,  $\rho = \infty$ ). Note that this is an additional assumption: the monotonicity of  $U$  alone does not guarantee it. The particles scattered through angles between  $\chi$  and  $\pi$  are then exactly those with impact parameters smaller than  $\rho(\chi)$ , so they fill a disc of area  $\pi\rho^2$  in the incident beam, and

$$\pi\rho^2(\chi) = \int_{\chi}^{\pi} \frac{d\sigma}{d\theta} d\theta.$$

Knowing the cross-section, this integral gives  $\rho(\chi)$ , which we invert to get the scattering angle  $\chi(\rho)$  as a function of the impact parameter. The problem is thus reduced to finding  $U(r)$  from  $\chi(\rho)$ .

The scattering angle in a central field is given by

$$\chi = \pi - 2 \int_{r_{min}}^{\infty} \frac{\rho dr}{r^2 \sqrt{1 - \frac{\rho^2}{r^2} - \frac{U(r)}{E}}}.$$

We now introduce the variables

$$s = \frac{1}{r}, \quad x = \frac{1}{\rho^2}, \quad w(s) = \sqrt{1 - \frac{U(1/s)}{E}},$$

and write  $\hat{\chi}(x) = \chi(x^{-1/2})$  for the same scattering angle expressed in terms of  $x$ . Since  $dr = -ds/s^2$ , we have  $\rho dr/r^2 = -\rho ds$ . When  $r$  goes from  $r_{min}$  to  $\infty$ ,  $s$  goes from  $s_0 = 1/r_{min}$  to 0, and the minus sign is absorbed by reversing the limits. Thus

$$\frac{\pi - \hat{\chi}(x)}{2} = \int_0^{s_0} \frac{\rho ds}{\sqrt{w^2(s) - \rho^2 s^2}} = \int_0^{s_0(x)} \frac{ds}{\sqrt{xw^2(s) - s^2}},$$

where we divided the numerator and the denominator by  $\rho = x^{-1/2}$ . The turning point  $s_0(x)$  is the zero of the radicand,

$$xw^2(s_0) - s_0^2 = 0.$$

Since  $U$  decreases with  $r$ , it increases with  $s$ , so  $w(s)$  decreases and  $s/w(s)$  increases with  $s$ . The radicand is therefore positive for  $0 \leq s < s_0(x)$ , and the condition  $s < s_0(x)$  is equivalent to  $x > s^2/w^2(s)$ .

This is an integral equation of Abel's type, which we solve as follows. Let  $A$  be an auxiliary parameter (denoted  $\alpha$  in the book). We divide both sides by  $\sqrt{A - x}$  and integrate over  $x$  from 0 to  $A$ ,

$$\frac{1}{2} \int_0^A \frac{\pi - \hat{\chi}(x)}{\sqrt{A - x}} dx = \int_0^A \int_0^{s_0(x)} \frac{ds dx}{\sqrt{(xw^2(s) - s^2)(A - x)}}.$$

The domain of integration on the right is  $0 < x < A$ ,  $0 < s < s_0(x)$ . From the previous remark, it can equally be described by  $0 < s < s_0(A)$  and  $s^2/w^2(s) < x < A$ . Reversing the order of integration and writing  $xw^2 - s^2 = w^2(x - s^2/w^2)$ , we get

$$\frac{1}{2} \int_0^A \frac{\pi - \hat{\chi}(x)}{\sqrt{A - x}} dx = \int_0^{s_0(A)} \frac{ds}{w(s)} \int_{s^2/w^2(s)}^A \frac{dx}{\sqrt{(x - s^2/w^2(s))(A - x)}}.$$

For the inner integral, let  $a = s^2/w^2$  and substitute  $x = a + (A - a)\sin^2 \varphi$ , so that  $dx = 2(A -$

a)  $\sin \varphi \cos \varphi \, d\varphi$  and  $\sqrt{(x-a)(A-x)} = (A-a) \sin \varphi \cos \varphi$ . Then

$$\int_a^A \frac{dx}{\sqrt{(x-a)(A-x)}} = \int_0^{\pi/2} 2 \, d\varphi = \pi,$$

independently of  $a$  and  $A$ . Hence

$$\frac{1}{2} \int_0^A \frac{\pi - \widehat{\chi}(x)}{\sqrt{A-x}} \, dx = \pi \int_0^{s_0(A)} \frac{ds}{w(s)}.$$

On the left, the first term gives  $\frac{1}{2}\pi \int_0^A (A-x)^{-1/2} \, dx = \pi\sqrt{A}$ . For the second term, we integrate by parts with  $dx/\sqrt{A-x} = -d(2\sqrt{A-x})$ ,

$$\begin{aligned} \int_0^A \frac{\widehat{\chi}(x) \, dx}{\sqrt{A-x}} &= \left[ -2\widehat{\chi}(x)\sqrt{A-x} \right]_0^A + 2 \int_0^A \widehat{\chi}'(x)\sqrt{A-x} \, dx \\ &= 2 \int_0^A \widehat{\chi}'(x)\sqrt{A-x} \, dx, \end{aligned}$$

since the boundary term vanishes at  $x = A$  and  $\widehat{\chi}(0) = 0$  (a particle with infinite impact parameter is not deflected). Therefore

$$\pi\sqrt{A} - \int_0^A \widehat{\chi}'(x)\sqrt{A-x} \, dx = \pi \int_0^{s_0(A)} \frac{ds}{w(s)}.$$

We differentiate this with respect to  $A$ . The integrand on the left vanishes at the upper limit  $x = A$ , so only the derivative of  $\sqrt{A-x}$  contributes,

$$\frac{\pi}{2\sqrt{A}} - \frac{1}{2} \int_0^A \frac{\widehat{\chi}'(x) \, dx}{\sqrt{A-x}} = \frac{\pi}{w(s_0)} \frac{ds_0}{dA}.$$

This holds for every  $A$ . Since  $A$  and  $s_0$  are related by  $Aw^2(s_0) = s_0^2$ , we may instead take  $s_0$  as the independent variable, rename it  $s$ , and put  $A = s^2/w^2$  with  $w = w(s)$ . Multiplying by  $dA = 2(s/w) \, d(s/w)$  and using  $\sqrt{A} = s/w$ , we get

$$\pi \, d\left(\frac{s}{w}\right) - \frac{s}{w} \, d\left(\frac{s}{w}\right) \int_0^{s^2/w^2} \frac{\widehat{\chi}'(x) \, dx}{\sqrt{s^2/w^2 - x}} = \frac{\pi \, ds}{w}.$$

Since

$$\pi \, d\left(\frac{s}{w}\right) - \frac{\pi \, ds}{w} = -\frac{\pi s \, dw}{w^2} = \frac{s}{w} (-\pi \, d \log w),$$

dividing by  $s/w$  gives

$$-\pi \, d \log w = d\left(\frac{s}{w}\right) \int_0^{s^2/w^2} \frac{\widehat{\chi}'(x) \, dx}{\sqrt{s^2/w^2 - x}}.$$

Let  $h = s/w$ , which increases from 0 with  $s$ . At  $s = 0$  (i.e.  $r = \infty$ ),  $U = 0$  and  $w = 1$ , so  $\log w = 0$ . Integrating from  $s = 0$ , and calling  $\eta$  the variable of integration for  $h$ ,

$$-\pi \log w = \int_0^h d\eta \int_0^{\eta^2} \frac{\widehat{\chi}'(x) \, dx}{\sqrt{\eta^2 - x}}.$$

The domain  $0 < \eta < h$ ,  $0 < x < \eta^2$  is also  $0 < x < h^2$ ,  $\sqrt{x} < \eta < h$ . Interchanging the integrations,

$$-\pi \log w = \int_0^{h^2} \widehat{\chi}'(x) \left[ \int_{\sqrt{x}}^h \frac{d\eta}{\sqrt{\eta^2 - x}} \right] dx = \int_0^{h^2} \widehat{\chi}'(x) \operatorname{arcosh} \left( \frac{h}{\sqrt{x}} \right) dx,$$

where we used  $\int d\eta/\sqrt{\eta^2 - x} = \operatorname{arcosh}(\eta/\sqrt{x})$  and  $\operatorname{arcosh} 1 = 0$ .

We now go back to the variables  $r$  and  $\rho$ . We have  $h = 1/(rw)$  and  $\sqrt{x} = 1/\rho$ , so  $h/\sqrt{x} = \rho/(rw)$ , and  $\widehat{\chi}'(x) dx = d\chi = (d\chi/d\rho) d\rho$ . The limit  $x = 0$  corresponds to  $\rho = \infty$  and  $x = h^2$  to  $\rho = rw$ . Hence

$$-\pi \log w = \int_{\infty}^{rw} \operatorname{arcosh} \left( \frac{\rho}{rw} \right) \frac{d\chi}{d\rho} d\rho = - \int_{rw}^{\infty} \operatorname{arcosh} \left( \frac{\rho}{rw} \right) \frac{d\chi}{d\rho} d\rho,$$

*i.e.*

$$w(r) = \exp \left\{ \frac{1}{\pi} \int_{rw(r)}^{\infty} \operatorname{arcosh} \left( \frac{\rho}{rw(r)} \right) \frac{d\chi}{d\rho} d\rho \right\}.$$

With the assumption adopted at the start, that  $\chi$  decreases monotonically with  $\rho$ , we have  $d\chi/d\rho < 0$ . The exponent is then negative and  $w < 1$ , *i.e.*  $U > 0$ , as expected for a repulsive field.

To obtain the second form, we integrate by parts in  $\rho$ , keeping  $r$  and  $w(r)$  fixed. Since

$$\frac{d}{d\rho} \operatorname{arcosh} \left( \frac{\rho}{rw} \right) = \frac{1}{\sqrt{\rho^2 - r^2 w^2}},$$

we have

$$\begin{aligned} \int_{rw}^{\infty} \operatorname{arcosh} \left( \frac{\rho}{rw} \right) \frac{d\chi}{d\rho} d\rho &= \left[ \chi(\rho) \operatorname{arcosh} \left( \frac{\rho}{rw} \right) \right]_{rw}^{\infty} \\ &\quad - \int_{rw}^{\infty} \frac{\chi(\rho) d\rho}{\sqrt{\rho^2 - r^2 w^2}}. \end{aligned}$$

Two conditions on the behaviour of  $\chi(\rho)$  at large  $\rho$  are needed here. **(a)** The boundary term vanishes at  $\rho = rw$  because  $\operatorname{arcosh} 1 = 0$ . At  $\rho \rightarrow \infty$ ,  $\operatorname{arcosh}(\rho/rw) \simeq \log(2\rho/rw)$  grows logarithmically, so the boundary term vanishes there only if  $\chi(\rho) \log \rho \rightarrow 0$ . **(b)** The remaining integral must converge. Its integrand behaves as  $\chi(\rho)/\rho$  at large  $\rho$ , so at infinity its convergence is equivalent to that of  $\int_{\infty}^{\infty} \chi(\rho) d\rho/\rho$ . Near the lower limit, the singularity  $1/\sqrt{\rho - rw}$  is integrable. Condition (a) alone does not guarantee (b): for example,  $\chi = 1/(\log \rho \log \log \rho)$  satisfies  $\chi \log \rho \rightarrow 0$ , but  $\int_{\infty}^{\infty} \chi d\rho/\rho$  diverges. A sufficient condition for both is  $\chi(\rho) = O(\rho^{-\varepsilon})$  with some  $\varepsilon > 0$ : then  $\chi \log \rho = O(\rho^{-\varepsilon} \log \rho) \rightarrow 0$ , and  $\int_{\infty}^{\infty} \rho^{-1-\varepsilon} d\rho$  converges. For instance, for a field decreasing as  $r^{-n}$  at large distances,  $\chi \propto \rho^{-n}$  for large  $\rho$  (see §20 Prob. 2). Under these conditions,

$$w(r) = \exp \left\{ -\frac{1}{\pi} \int_{rw(r)}^{\infty} \frac{\chi(\rho) d\rho}{\sqrt{\rho^2 - r^2 w^2(r)}} \right\}.$$

Both formulas give  $w$  only implicitly, since  $w$  also appears in the lower limit. However, the right-hand sides depend on  $r$  and  $w$  only through the product  $rw$ . They therefore give  $w$  as an explicit function of  $rw$ , from which  $w(r)$  follows, and then

$$U(r) = E [1 - w^2(r)].$$

The derivation covered all  $s$  between 0 and the turning points  $s_0$ . The smallest distance reached is that of the head-on collision ( $\rho = 0$ ),  $r_{\min}$ , for which  $U(r_{\min}) = E$  and  $w = 0$ . The potential is thus determined in the accessible region  $r > r_{\min}$ . The scattering data give no information on the interior region  $r < r_{\min}$ , where  $U > E$ . In summary, we get

#### ANSWER TO PROBLEM 7.

$$\begin{aligned} w(r) &= \exp \left\{ \frac{1}{\pi} \int_{rw(r)}^{\infty} \operatorname{arcosh} \left( \frac{\rho}{rw(r)} \right) \frac{d\chi}{d\rho} d\rho \right\} = \exp \left\{ -\frac{1}{\pi} \int_{rw(r)}^{\infty} \frac{\chi(\rho) d\rho}{\sqrt{\rho^2 - r^2 w^2(r)}} \right\}, \\ w(r) &= \sqrt{1 - \frac{U(r)}{E}}, \quad U(r) = E [1 - w^2(r)], \quad r > r_{\min} \end{aligned}$$

## §19 RUTHERFORD'S FORMULA

## PROBLEM 1.

Find the effective cross-section for scattering in a field  $U = \alpha/r^2$  ( $\alpha > 0$ ).

SOLUTION: Let the particle of mass  $m$  come from infinity with velocity  $v_\infty$  and impact parameter  $\rho$ . It is convenient to introduce the positive length  $a$  defined by

$$a^2 = \frac{2\alpha}{mv_\infty^2}.$$

With  $U(r) = \alpha/r^2$ , we have  $2U/(mv_\infty^2) = a^2/r^2$ , so the angle between the direction of closest approach and the asymptote of the trajectory is

$$\phi_0 = \int_{r_{\min}}^{\infty} \frac{\rho \, dr}{r^2 \sqrt{1 - \frac{\rho^2}{r^2} - \frac{a^2}{r^2}}} = \int_{r_{\min}}^{\infty} \frac{\rho \, dr}{r^2 \sqrt{1 - (\rho^2 + a^2)/r^2}}.$$

The distance of closest approach is the zero of the radicand,

$$r_{\min} = \sqrt{\rho^2 + a^2}.$$

We substitute  $u = 1/r$ , so that  $dr/r^2 = -du$ . When  $r$  goes from  $r_{\min}$  to  $\infty$ ,  $u$  goes from  $1/\sqrt{\rho^2 + a^2}$  to 0, and the minus sign is absorbed by reversing the limits,

$$\begin{aligned} \phi_0 &= \rho \int_0^{1/\sqrt{\rho^2 + a^2}} \frac{du}{\sqrt{1 - (\rho^2 + a^2)u^2}} \\ &= \frac{\rho}{\sqrt{\rho^2 + a^2}} \left[ \arcsin \left( \sqrt{\rho^2 + a^2} u \right) \right]_0^{1/\sqrt{\rho^2 + a^2}} = \frac{\pi \rho}{2\sqrt{\rho^2 + a^2}}. \end{aligned}$$

The scattering angle is therefore

$$\chi = \pi - 2\phi_0 = \pi \left( 1 - \frac{\rho}{\sqrt{\rho^2 + a^2}} \right) = \pi \left[ 1 - \left( 1 + \frac{2\alpha}{mv_\infty^2 \rho^2} \right)^{-1/2} \right].$$

Since  $\rho/\sqrt{\rho^2 + a^2}$  increases from 0 to 1 as  $\rho$  goes from 0 to  $\infty$ ,  $\chi$  decreases monotonically from  $\pi$  for a head-on collision to 0 as  $\rho \rightarrow \infty$ . Thus each angle  $0 < \chi < \pi$  corresponds to a single impact parameter, and we can invert this relation. Writing it as

$$\frac{\rho}{\sqrt{\rho^2 + a^2}} = \frac{\pi - \chi}{\pi},$$

and squaring, we get  $\pi^2 \rho^2 = (\pi - \chi)^2 (\rho^2 + a^2)$ , i.e.

$$\rho^2 [\pi^2 - (\pi - \chi)^2] = a^2 (\pi - \chi)^2 \Rightarrow \rho^2 = a^2 \frac{(\pi - \chi)^2}{\chi(2\pi - \chi)},$$

where we used  $\pi^2 - (\pi - \chi)^2 = \chi(2\pi - \chi)$ . To differentiate, let  $y = \pi - \chi$ , so that  $\rho^2 = a^2 y^2 / (\pi^2 - y^2)$  and

$$\frac{d(\rho^2)}{dy} = a^2 \frac{2y(\pi^2 - y^2) + 2y^3}{(\pi^2 - y^2)^2} = \frac{2\pi^2 a^2 y}{(\pi^2 - y^2)^2}.$$

Since  $dy = -d\chi$ ,

$$\frac{d(\rho^2)}{d\chi} = -\frac{2\pi^2 a^2 (\pi - \chi)}{\chi^2 (2\pi - \chi)^2}.$$

The particles scattered into the range  $d\chi$  are those passing through an annulus of radius  $\rho$  and width  $|d\rho|$  in the incident beam, so

$$d\sigma = 2\pi\rho \left| \frac{d\rho}{d\chi} \right| d\chi = \pi \left| \frac{d(\rho^2)}{d\chi} \right| d\chi.$$

In terms of the element of solid angle  $do = 2\pi \sin \chi \, d\chi$ , this becomes

$$d\sigma = \frac{1}{2 \sin \chi} \left| \frac{d(\rho^2)}{d\chi} \right| do = \pi^2 a^2 \frac{\pi - \chi}{\chi^2 (2\pi - \chi)^2 \sin \chi} do.$$

This is a differential cross-section. Its integral over all angles diverges at  $\chi = 0$ , since particles with

arbitrarily large impact parameters are still deflected, by arbitrarily small angles. As a check, for  $\rho \gg a$  we have  $\rho/\sqrt{\rho^2 + a^2} \simeq 1 - a^2/(2\rho^2)$ , hence

$$\chi \simeq \frac{\pi a^2}{2\rho^2} = \frac{\pi\alpha}{mv_\infty^2 \rho^2}, \quad \frac{d\sigma}{d\alpha} \simeq \pi^2 a^2 \frac{\pi}{4\pi^2 \chi^3} = \frac{\pi\alpha}{2mv_\infty^2 \chi^3}.$$

The same small-angle result follows directly from  $\rho^2 \simeq \pi\alpha/(mv_\infty^2 \chi)$ , which gives  $d\sigma = \pi|d(\rho^2)/d\chi| d\chi = \pi^2 \alpha d\chi/(mv_\infty^2 \chi^2)$  with  $d\alpha \simeq 2\pi\chi d\chi$ . Substituting  $a^2$ , we thus get

ANSWER TO PROBLEM 1.

$$\chi = \pi \left[ 1 - \left( 1 + \frac{2\alpha}{mv_\infty^2 \rho^2} \right)^{-1/2} \right],$$

$$d\sigma = \frac{2\pi^2 \alpha}{mv_\infty^2} \frac{\pi - \chi}{\chi^2 (2\pi - \chi)^2} \frac{d\alpha}{\sin \chi}, \quad 0 < \chi < \pi$$

## PROBLEM 2.

Find the effective cross-section for scattering by a spherical “potential well” of radius  $a$  and “depth”  $U_0$  (*i.e.* a field with  $U = 0$  for  $r > a$  and  $U = -U_0$  for  $r < a$ ).

SOLUTION: Let  $m$  be the mass of the particle and  $v_\infty$  its velocity at infinity. Since  $U$  is constant outside the sphere, the particle moves with the constant velocity  $v_\infty$  until it reaches  $r = a$ . Inside the well,  $U$  is again constant, so the particle moves with a constant velocity  $v_{\text{in}}$  given by the conservation of energy,

$$\frac{1}{2}mv_{\text{in}}^2 - U_0 = \frac{1}{2}mv_\infty^2.$$

As in the book, we define the ratio

$$n = \frac{v_{\text{in}}}{v_\infty} = \sqrt{1 + \frac{2U_0}{mv_\infty^2}} > 1.$$

The trajectory is therefore made of straight segments, and its direction changes only at the boundary  $r = a$ , where the particle receives an impulse directed along the radius. This is analogous to the refraction of a light ray by a sphere with refractive index  $n$ .

The geometry of the path is shown in Fig. 21 of the book. Let  $\alpha$  be the angle of incidence and  $\beta$  the angle of refraction, both measured from the normal to the sphere (*i.e.* the radius) at the point of entry. At  $r = a$ , the angular momentum is conserved (the force is radial), so the component of the velocity perpendicular to the radius is the same on both sides,

$$mv_\infty a \sin \alpha = mv_{\text{in}} a \sin \beta \Rightarrow \frac{\sin \alpha}{\sin \beta} = n.$$

Particles with  $\rho > a$  do not touch the sphere and are not deflected. For  $\rho < a$ , the impact parameter is the distance from the centre to the incident line, so

$$\rho = a \sin \alpha.$$

At the entry, the direction of motion turns towards the centre by an angle  $\alpha - \beta$ . Inside the sphere, the particle moves along a chord, which makes the same angle  $\beta$  with the radii at both of its ends. By the symmetry of the path with respect to the radius bisecting this chord, the particle leaves the sphere at the angle  $\alpha$  with the normal and is again deflected by  $\alpha - \beta$  in the same sense. The total angle of deflection is thus

$$\chi = 2(\alpha - \beta).$$

To express  $\rho$  in terms of  $\chi$ , let  $q = \chi/2$ , so that  $\beta = \alpha - q$ . The law of refraction becomes

$$\sin(\alpha - q) = \sin \alpha \cos q - \cos \alpha \sin q = \frac{\sin \alpha}{n}.$$

Dividing by  $\sin \alpha$ ,

$$\cos q - \cot \alpha \sin q = \frac{1}{n} \Rightarrow \cot \alpha = \frac{n \cos q - 1}{n \sin q}.$$

Using  $\sin^2 \alpha = 1/(1 + \cot^2 \alpha)$ , we get

$$\sin^2 \alpha = \frac{n^2 \sin^2 q}{n^2 \sin^2 q + (n \cos q - 1)^2} = \frac{n^2 \sin^2 q}{n^2 + 1 - 2n \cos q},$$

since  $n^2 \sin^2 q + n^2 \cos^2 q - 2n \cos q + 1 = n^2 + 1 - 2n \cos q$ . Therefore

$$\rho^2 = a^2 \sin^2 \alpha = \frac{a^2 n^2 \sin^2(\chi/2)}{n^2 + 1 - 2n \cos(\chi/2)}.$$

Let  $D = n^2 + 1 - 2n \cos q$ , so that  $dD/dq = 2n \sin q$ . Since  $dq/d\chi = 1/2$ , the quotient rule gives

$$\frac{d(\rho^2)}{d\chi} = \frac{a^2 n^2}{2} \frac{2 \sin q \cos q D - \sin^2 q \cdot 2n \sin q}{D^2} = \frac{a^2 n^2 \sin q [\cos q D - n \sin^2 q]}{D^2}.$$

The bracket factorizes as

$$\begin{aligned}
\cos q D - n \sin^2 q &= n^2 \cos q + \cos q - 2n \cos^2 q - n + n \cos^2 q \\
&= n^2 \cos q - n \cos^2 q - n + \cos q \\
&= (n \cos q - 1)(n - \cos q).
\end{aligned}$$

We now check that  $\chi$  increases with  $\rho$ . As  $\rho$  increases from 0 to  $a$ ,  $\alpha$  increases from 0 to  $\pi/2$ , and since  $\sin \beta = \sin \alpha/n$ ,

$$\frac{d\chi}{d\alpha} = 2 \left( 1 - \frac{d\beta}{d\alpha} \right) = 2 \left( 1 - \frac{\cos \alpha}{n \cos \beta} \right) = 2 \left( 1 - \frac{\cos \alpha}{\sqrt{n^2 - \sin^2 \alpha}} \right) > 0,$$

because  $n^2 - \sin^2 \alpha > 1 - \sin^2 \alpha = \cos^2 \alpha$ . The relation between  $\rho$  and  $\chi$  is thus one-to-one, and  $d(\rho^2)/d\chi > 0$ . The particles deflected into the range  $d\chi$  pass through an annulus of area

$$d\sigma = \pi \frac{d(\rho^2)}{d\chi} d\chi = \frac{1}{2 \sin \chi} \frac{d(\rho^2)}{d\chi} d\chi,$$

where we used  $d\phi = 2\pi \sin \chi d\chi$ . With  $\sin \chi = 2 \sin q \cos q$ , the factor  $\sin q$  cancels, and

$$d\sigma = \frac{a^2 n^2 (n \cos q - 1)(n - \cos q)}{4 \cos q D^2} d\phi.$$

Let us find the range of  $\chi$ . For  $\rho = 0$ , the particle crosses the sphere along a diameter, so  $\alpha = \beta = 0$  and  $\chi = 0$ . As  $\rho \rightarrow a$  from below, the particle grazes the sphere, with

$$\alpha \rightarrow \frac{\pi}{2}, \quad \beta \rightarrow \arcsin \frac{1}{n}.$$

Hence

$$\chi_{max} = 2 \left( \frac{\pi}{2} - \arcsin \frac{1}{n} \right) = 2 \arccos \frac{1}{n}, \quad \cos \frac{\chi_{max}}{2} = \frac{1}{n}.$$

The formula for  $d\sigma$  applies for  $0 < \chi < \chi_{max}$ . Accordingly, the factor  $n \cos(\chi/2) - 1$  is positive in this range and vanishes at  $\chi = \chi_{max}$ . Particles with  $\rho > a$  are not deflected and do not contribute to the scattering.

Finally, the total cross-section for scattering is obtained directly from the area of the annuli,

$$\sigma = \int_0^{\chi_{max}} d\sigma = \pi [\rho^2(\chi_{max}) - \rho^2(0)].$$

At  $\chi = \chi_{max}$ ,  $\cos q = 1/n$ , so  $\sin^2 q = (n^2 - 1)/n^2$  and  $D = n^2 - 1$ , which gives  $\rho^2(\chi_{max}) = a^2$ . Since  $\rho^2(0) = 0$ ,  $\sigma = \pi a^2$ . This is simply the cross-sectional area of the sphere: every particle which hits the sphere is deflected. We thus get

#### ANSWER TO PROBLEM 2.

$$\begin{aligned}
d\sigma &= \frac{a^2 n^2 [n \cos(\chi/2) - 1] [n - \cos(\chi/2)]}{4 \cos(\chi/2) [n^2 + 1 - 2n \cos(\chi/2)]^2} d\phi, & n &= \sqrt{1 + \frac{2U_0}{mv_\infty^2}}, \\
0 < \chi < \chi_{max} &= 2 \arccos \frac{1}{n}, & \sigma &= \pi a^2
\end{aligned}$$

## §20 SMALL-ANGLE SCATTERING

## PROBLEM 1.

Derive formula (20.3) from (18.4).

SOLUTION: Consider the relative motion, with reduced mass  $m$  and relative velocity at infinity  $v_\infty$ . Formula (18.4) of the book gives

$$\phi_0 = \int_{r_{\min}}^{\infty} \frac{\rho \, dr}{r^2 \sqrt{1 - \frac{\rho^2}{r^2} - \frac{2U(r)}{mv_\infty^2}}}, \quad \chi = \pi - 2\phi_0.$$

For a weak field, it is tempting to expand the integrand in powers of  $U$ . However, the integrand behaves as  $1/\sqrt{r - r_{\min}}$  near the turning point. Expanding it gives terms like  $U/(1 - \rho^2/r^2)^{3/2}$ , which are not integrable at  $r = \rho$ . These divergences are spurious: they only come from expanding near the point where the radicand vanishes.

To avoid this, we write  $\phi_0$  as the derivative of an integral whose integrand vanishes at the turning point. Since this integral does not converge at infinity, we first introduce a finite upper limit  $R$ ,

$$J_R(\rho) = \int_{r_{\min}}^R \sqrt{1 - \frac{\rho^2}{r^2} - \frac{2U(r)}{mv_\infty^2}} \, dr.$$

We differentiate with respect to  $\rho$  at fixed  $R$ . The lower limit  $r_{\min}$  depends on  $\rho$ , but its contribution is the integrand at  $r = r_{\min}$  times  $dr_{\min}/d\rho$ , which vanishes since the square root is zero there. Hence

$$\frac{\partial J_R}{\partial \rho} = - \int_{r_{\min}}^R \frac{\rho \, dr}{r^2 \sqrt{1 - \frac{\rho^2}{r^2} - \frac{2U(r)}{mv_\infty^2}}} \Rightarrow \phi_{0,R} = - \frac{\partial J_R}{\partial \rho},$$

where  $\phi_{0,R}$  is the integral (18.4) cut off at  $r = R$ . We keep  $R$  finite until after the differentiation, since  $J_R$  itself grows like  $R$  and has no limit as  $R \rightarrow \infty$ .

We now expand  $J_R$  to first order in  $U$ . Pointwise,

$$\sqrt{1 - \frac{\rho^2}{r^2} - \frac{2U}{mv_\infty^2}} \simeq \sqrt{1 - \frac{\rho^2}{r^2}} - \frac{U}{mv_\infty^2 \sqrt{1 - \rho^2/r^2}} = \sqrt{1 - \frac{\rho^2}{r^2}} - \frac{1}{mv_\infty^2} \frac{U(r) r}{\sqrt{r^2 - \rho^2}}.$$

This expansion is not uniform near the turning point, but it can be integrated. To see why, regard  $J_R$  as a function of the strength of the field. Its first-order variation contains a term from the shift of the lower limit, from  $\rho$  (the turning point when  $U = 0$ ) to  $r_{\min}$ . This term is proportional to the unperturbed integrand at its lower endpoint, which vanishes. The other term is the integral of the first-order variation of the integrand, which has only an integrable singularity  $1/\sqrt{r - \rho}$ . To first order, we can therefore take the lower limit at  $\rho$ ,

$$J_R \simeq \int_{\rho}^R \sqrt{1 - \frac{\rho^2}{r^2}} \, dr - \frac{1}{mv_\infty^2} \int_{\rho}^R \frac{U(r) r \, dr}{\sqrt{r^2 - \rho^2}} = \int_{\rho}^R \sqrt{1 - \frac{\rho^2}{r^2}} \, dr - \frac{K_R(\rho)}{mv_\infty^2},$$

where

$$K_R(\rho) = \int_{\rho}^R \frac{U(r) r \, dr}{\sqrt{r^2 - \rho^2}}.$$

The free term can be integrated directly,

$$\int_{\rho}^R \sqrt{1 - \frac{\rho^2}{r^2}} \, dr = \int_{\rho}^R \frac{\sqrt{r^2 - \rho^2}}{r} \, dr = \left[ \sqrt{r^2 - \rho^2} - \rho \arccos \frac{\rho}{r} \right]_{\rho}^R = \sqrt{R^2 - \rho^2} - \rho \arccos \frac{\rho}{R},$$

as can be checked by differentiating the bracket with respect to  $r$ ,

$$\frac{r}{\sqrt{r^2 - \rho^2}} - \frac{\rho^2}{r \sqrt{r^2 - \rho^2}} = \frac{\sqrt{r^2 - \rho^2}}{r}.$$

Its derivative with respect to  $\rho$  is

$$\frac{\partial}{\partial \rho} \left( \sqrt{R^2 - \rho^2} - \rho \arccos \frac{\rho}{R} \right) = -\frac{\rho}{\sqrt{R^2 - \rho^2}} - \arccos \frac{\rho}{R} + \frac{\rho}{\sqrt{R^2 - \rho^2}} = -\arccos \frac{\rho}{R}.$$

Consequently,

$$\phi_{0,R} \simeq \arccos \frac{\rho}{R} + \frac{1}{mv_\infty^2} \frac{\partial K_R}{\partial \rho}.$$

We cannot differentiate  $K_R$  directly under the integral sign, because its integrand is singular at the lower limit  $r = \rho$ . We therefore integrate by parts first, using

$$\frac{r \, dr}{\sqrt{r^2 - \rho^2}} = d\sqrt{r^2 - \rho^2},$$

which gives

$$K_R = \left[ U(r) \sqrt{r^2 - \rho^2} \right]_\rho^R - \int_\rho^R U'(r) \sqrt{r^2 - \rho^2} \, dr = U(R) \sqrt{R^2 - \rho^2} - \int_\rho^R U'(r) \sqrt{r^2 - \rho^2} \, dr.$$

The boundary term at  $r = \rho$  vanishes since  $\sqrt{r^2 - \rho^2} = 0$  there. The new integrand is finite at  $r = \rho$ , and it vanishes there. Hence, when we differentiate with respect to  $\rho$ , the variation of the lower limit again gives no contribution, and

$$\frac{\partial K_R}{\partial \rho} = -\frac{\rho U(R)}{\sqrt{R^2 - \rho^2}} + \rho \int_\rho^R \frac{U'(r) \, dr}{\sqrt{r^2 - \rho^2}}.$$

Now we take  $R \rightarrow \infty$ . Then  $\arccos(\rho/R) \rightarrow \pi/2$ , and the first term vanishes since  $U(R) \rightarrow 0$  and  $\rho/\sqrt{R^2 - \rho^2} \rightarrow 0$ . Assuming the remaining integral converges (*i.e.* the force decreases fast enough), we get

$$\phi_0 \simeq \frac{\pi}{2} + \frac{\rho}{mv_\infty^2} \int_\rho^\infty \frac{U'(r) \, dr}{\sqrt{r^2 - \rho^2}},$$

and therefore

$$\chi = \pi - 2\phi_0 \simeq -\frac{2\rho}{mv_\infty^2} \int_\rho^\infty \frac{U'(r) \, dr}{\sqrt{r^2 - \rho^2}}.$$

For a repulsive field,  $U' < 0$  and  $\chi > 0$ , as it should be. This result is valid only if the deflection is small, *i.e.* if the field is weak along the whole path ( $|U| \ll \frac{1}{2}mv_\infty^2$  for  $r \gtrsim \rho$ ). Under this condition, the first-order expansion is justified.

The angle  $\chi$  is the scattering angle in the centre-of-mass system. Formula (20.3) gives instead  $\theta_1$ , the deflection of the incident particle in the laboratory system, where the particle  $m_2$  is initially at rest. From formula (17.4) of the book, for small angles

$$\tan \theta_1 = \frac{m_2 \sin \chi}{m_1 + m_2 \cos \chi} \Rightarrow \theta_1 \simeq \frac{m_2}{m_1 + m_2} \chi = \frac{m}{m_1} \chi, \quad m = \frac{m_1 m_2}{m_1 + m_2}.$$

Multiplying  $\chi$  by  $m/m_1$ , the reduced mass cancels. We thus get

#### ANSWER TO PROBLEM 1.

$$\begin{aligned} \theta_1 &\simeq -\frac{2\rho}{m_1 v_\infty^2} \int_\rho^\infty \frac{dU}{dr} \frac{dr}{\sqrt{r^2 - \rho^2}}, \\ \chi &\simeq -\frac{2\rho}{mv_\infty^2} \int_\rho^\infty \frac{dU}{dr} \frac{dr}{\sqrt{r^2 - \rho^2}} \end{aligned}$$

## PROBLEM 2.

Determine the effective cross-section for small-angle scattering in a field  $U = \alpha/r^n$  ( $n > 0$ ).

SOLUTION: We take  $\alpha > 0$  (a repulsive field). Let  $m_2$  be the mass of the scattering particle, initially at rest, and

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

the reduced mass. As shown in §20 Prob. 1, when the scattering angle  $\chi$  in the centre-of-mass system is small ( $\chi \ll 1$ ), the deflection of the incident particle of mass  $m_1$  is  $\theta_1 \simeq (m/m_1)\chi$ , and it is given by formula (20.3) of the book,

$$\theta_1 \simeq -\frac{2\rho}{m_1 v_\infty^2} \int_\rho^\infty \frac{dU}{dr} \frac{dr}{\sqrt{r^2 - \rho^2}}.$$

Since

$$\frac{dU}{dr} = -\frac{n\alpha}{r^{n+1}},$$

the deflection is

$$\theta_1 \simeq \frac{2n\alpha\rho}{m_1 v_\infty^2} \int_\rho^\infty \frac{dr}{r^{n+1} \sqrt{r^2 - \rho^2}}.$$

To evaluate the integral, we substitute

$$u = \frac{\rho^2}{r^2}, \quad r = \rho u^{-1/2}, \quad dr = -\frac{\rho}{2} u^{-3/2} du.$$

When  $r$  goes from  $\rho$  to  $\infty$ ,  $u$  goes from 1 to 0, and the minus sign of  $dr$  is absorbed by reversing the limits. We also have

$$r^{n+1} = \rho^{n+1} u^{-(n+1)/2}, \quad \sqrt{r^2 - \rho^2} = \rho \sqrt{\frac{1}{u} - 1} = \rho u^{-1/2} (1 - u)^{1/2}.$$

Collecting the powers of  $u$ ,  $-\frac{3}{2} + \frac{n+1}{2} + \frac{1}{2} = \frac{n-1}{2}$ , so

$$\int_\rho^\infty \frac{dr}{r^{n+1} \sqrt{r^2 - \rho^2}} = \frac{1}{2\rho^{n+1}} \int_0^1 u^{(n-1)/2} (1-u)^{-1/2} du.$$

As in §11 Prob. 2, this is the Beta function  $B(p, q) = \int_0^1 u^{p-1} (1-u)^{q-1} du = \Gamma(p)\Gamma(q)/\Gamma(p+q)$  with  $p = (n+1)/2$  and  $q = 1/2$ . Since  $\Gamma(1/2) = \sqrt{\pi}$ ,

$$\int_\rho^\infty \frac{dr}{r^{n+1} \sqrt{r^2 - \rho^2}} = \frac{1}{2\rho^{n+1}} B\left(\frac{n+1}{2}, \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2\rho^{n+1}} \frac{\Gamma((n+1)/2)}{\Gamma((n+2)/2)}.$$

Using  $\Gamma((n+2)/2) = \frac{n}{2}\Gamma(n/2)$ , the factor  $n$  cancels and

$$\theta_1 \simeq \frac{2\sqrt{\pi}\alpha}{m_1 v_\infty^2 \rho^n} \frac{\Gamma((n+1)/2)}{\Gamma(n/2)}.$$

Let us define

$$C_n = \frac{2\sqrt{\pi}\alpha}{m_1 v_\infty^2} \frac{\Gamma((n+1)/2)}{\Gamma(n/2)},$$

so that  $\theta_1 \simeq C_n/\rho^n$ . The deflection decreases monotonically with  $\rho$ , and the relation can be inverted,

$$\rho \simeq \left(\frac{C_n}{\theta_1}\right)^{1/n}, \quad \left|\frac{d\rho}{d\theta_1}\right| = \frac{1}{n} C_n^{1/n} \theta_1^{-1/n-1} = \frac{\rho}{n\theta_1}.$$

The particles deflected into the range  $d\theta_1$  pass through an annulus of area  $2\pi\rho|d\rho|$  in the incident beam. For small angles, the element of solid angle is

$$d\Omega_1 = 2\pi \sin \theta_1 d\theta_1 \simeq 2\pi \theta_1 d\theta_1,$$

so

$$\frac{d\sigma}{d\Omega_1} \simeq \frac{\rho}{\theta_1} \left|\frac{d\rho}{d\theta_1}\right| = \frac{\rho^2}{n\theta_1^2} = \frac{1}{n} \frac{C_n^{2/n}}{\theta_1^{2+2/n}}.$$

This result requires weak scattering in the centre-of-mass system,  $\chi \simeq (m_1/m)\theta_1 \ll 1$ . Multiplying

$\theta_1 \simeq C_n/\rho^n$  by  $m_1/m$ , the corresponding centre-of-mass deflection is

$$\chi \simeq \frac{2\sqrt{\pi} \Gamma((n+1)/2)}{\Gamma(n/2)} \frac{\alpha/\rho^n}{mv_\infty^2},$$

whose numerical coefficient is of order unity for  $n$  of order unity. The condition  $\chi \ll 1$  is thus the weak-field condition

$$\frac{\alpha}{\rho^n} \ll \frac{1}{2}mv_\infty^2,$$

*i.e.* the potential energy at the distance of closest approach  $r \simeq \rho$  must be small compared with the energy of the relative motion. Since  $m \leq m_1$ , it implies  $\theta_1 \ll 1$ , which justifies  $do_1 \simeq 2\pi\theta_1 d\theta_1$ . However, the converse does not hold for arbitrary masses. For instance, if  $m_1 > m_2$ , the laboratory angle never exceeds  $\arcsin(m_2/m_1)$ , so it can be small even when  $\chi$  is not. The two conditions coincide only when  $m_2 \gg m_1$ , where  $m \simeq m_1$  and  $\theta_1 \simeq \chi$ . The result must not be extended to arbitrary angles. Note also that  $d\sigma$  diverges as  $\theta_1 \rightarrow 0$ , so the total cross-section is infinite unless the angles are cut off.

As a check, for  $n = 1$  we have  $\Gamma(1)/\Gamma(1/2) = 1/\sqrt{\pi}$ , so  $C_1 = 2\alpha/(m_1v_\infty^2)$  and

$$\frac{d\sigma}{do_1} \simeq \frac{4\alpha^2}{m_1^2v_\infty^4\theta_1^4},$$

which is the small-angle limit of the Rutherford formula. For  $n = 2$ ,  $\Gamma(3/2)/\Gamma(1) = \sqrt{\pi}/2$ , so  $C_2 = \pi\alpha/(m_1v_\infty^2)$  and

$$\frac{d\sigma}{do_1} \simeq \frac{\pi\alpha}{2m_1v_\infty^2\theta_1^3},$$

in agreement with the small-angle limit found in §19 Prob. 1 for a fixed centre ( $m_1 = m$ ,  $\theta_1 = \chi$ ). We thus get

#### ANSWER TO PROBLEM 2.

$$\begin{aligned} \theta_1 &\simeq \frac{2\sqrt{\pi} \Gamma((n+1)/2)}{\Gamma(n/2)} \frac{\alpha}{m_1v_\infty^2\rho^n}, \\ d\sigma &\simeq \frac{1}{n} \left[ \frac{2\sqrt{\pi} \Gamma((n+1)/2)}{\Gamma(n/2)} \frac{\alpha}{m_1v_\infty^2} \right]^{2/n} \frac{do_1}{\theta_1^{2+2/n}}, \\ \text{valid for } \frac{\alpha}{\rho^n} &\ll \frac{1}{2}mv_\infty^2, \quad m = \frac{m_1m_2}{m_1+m_2} \end{aligned}$$

## CHAPTER 5.

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### SMALL OSCILLATIONS

## §21 FREE OSCILLATIONS IN ONE DIMENSION

## PROBLEM 1.

Express the amplitude and initial phase of the oscillations in terms of the initial co-ordinate  $x_0$  and velocity  $v_0$ .

SOLUTION: The general solution of  $\ddot{x} + \omega^2 x = 0$ , with angular frequency  $\omega > 0$ , that satisfies  $x(0) = x_0$  and  $\dot{x}(0) = v_0$  is

$$x(t) = x_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t.$$

We write the same harmonic motion as

$$x(t) = a \cos(\omega t + \alpha) = a \cos \alpha \cos \omega t - a \sin \alpha \sin \omega t, \quad a \geq 0.$$

Comparing the coefficients of  $\cos \omega t$  and  $\sin \omega t$ , we get

$$a \cos \alpha = x_0, \quad a \sin \alpha = -\frac{v_0}{\omega}.$$

These are the relations  $x_0 = a \cos \alpha$  and  $v_0 = -a\omega \sin \alpha$  obtained by evaluating  $x$  and  $\dot{x}$  directly at  $t = 0$ . Squaring them and adding gives

$$x_0^2 + \frac{v_0^2}{\omega^2} = a^2 (\cos^2 \alpha + \sin^2 \alpha) = a^2.$$

For a nonzero amplitude, the phase is fixed modulo  $2\pi$  by

$$\cos \alpha = \frac{x_0}{a}, \quad \sin \alpha = -\frac{v_0}{a\omega}.$$

Thus,  $\tan \alpha = -v_0/(\omega x_0)$  when  $x_0 \neq 0$ , with the quadrant determined by the signs of the sine and cosine. The two-argument arctangent  $\text{atan2}(Y, X)$  returns the angle whose cosine and sine have the signs of  $X$  and  $Y$  respectively, and so selects the correct quadrant. If  $x_0 = v_0 = 0$ , the particle remains at equilibrium and the phase is arbitrary.

## ANSWER TO PROBLEM 1.

$$a = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}, \quad \alpha = \text{atan2}\left(-\frac{v_0}{\omega}, x_0\right) \pmod{2\pi}$$

## PROBLEM 2.

Find the ratio of frequencies  $\omega$  and  $\omega'$  of the oscillations of two diatomic molecules consisting of atoms of different isotopes, the masses of the atoms being  $m_1, m_2$  and  $m'_1, m'_2$ .

SOLUTION: Within the harmonic approximation, isotopic substitution changes the masses while leaving the interatomic potential, and hence its curvature  $k$  at equilibrium, unchanged.

Let  $u_1$  and  $u_2$  be the displacements of the atoms from their equilibrium positions along the molecular axis, with atom 2 on the positive side of atom 1. We define the centre-of-mass displacement  $X$  and the change  $q$  in interatomic separation from equilibrium by

$$X = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}, \quad q = u_2 - u_1.$$

Inverting these relations gives

$$u_1 = X - \frac{m_2}{m_1 + m_2} q, \quad u_2 = X + \frac{m_1}{m_1 + m_2} q.$$

Substituting them into the kinetic energy, the cross terms in  $\dot{X}\dot{q}$  cancel, and we get

$$T = \frac{1}{2} m_1 \dot{u}_1^2 + \frac{1}{2} m_2 \dot{u}_2^2 = \frac{1}{2} (m_1 + m_2) \dot{X}^2 + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} \dot{q}^2.$$

The first term describes the free translation of the molecule as a whole, and the second describes its vibration. The potential energy depends only on the interatomic separation, and hence only on  $q$ ; in the harmonic approximation, it is  $\frac{1}{2} k q^2$ . Dropping the translational term, the vibrational Lagrangian is

$$L_{\text{vib}} = \frac{1}{2} \mu \dot{q}^2 - \frac{1}{2} k q^2, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}.$$

The equation of motion is

$$\mu \ddot{q} + k q = 0,$$

so that

$$\omega = \sqrt{\frac{k}{\mu}}.$$

For the isotopically substituted molecule,

$$\mu' = \frac{m'_1 m'_2}{m'_1 + m'_2}, \quad \omega' = \sqrt{\frac{k}{\mu'}}.$$

Taking the ratio eliminates the common force constant.

## ANSWER TO PROBLEM 2.

$$\frac{\omega'}{\omega} = \sqrt{\frac{\mu}{\mu'}} = \sqrt{\frac{m_1 m_2 (m'_1 + m'_2)}{m'_1 m'_2 (m_1 + m_2)}}$$

**PROBLEM 3.**

Find the frequency of oscillations of a particle of mass  $m$  which is free to move along a line and is attached to a spring whose other end is fixed at a point  $A$  (Fig. 22) at a distance  $l$  from the line. A force  $F$  is required to extend the spring to length  $l$ .

**SOLUTION:** Let  $x$  be the displacement along the line from the point nearest the fixed end  $A$ . The length of the spring is

$$d(x) = \sqrt{l^2 + x^2}.$$

Here,  $l$  is the length of the spring in the constrained equilibrium configuration  $x = 0$ , where the spring is already stretched and carries a tension  $F > 0$ ; it is not the natural length of the spring. For  $|x| \ll l$ , the additional extension from this equilibrium length is

$$\delta l = d(x) - l = l \left( \sqrt{1 + \frac{x^2}{l^2}} - 1 \right) \simeq \frac{x^2}{2l}.$$

Let  $V(d)$  denote the potential energy of the spring. The force required to hold it at length  $l$  is  $F = V'(l) > 0$ . Expanding about that length and dropping the constant  $V(l)$ , we get

$$U(x) = V(l + \delta l) - V(l) = F \delta l + \frac{1}{2} V''(l) (\delta l)^2 + \dots$$

Since  $\delta l$  is already quadratic in  $x$ , its square contributes only at fourth order. Thus, the harmonic approximation gives

$$U(x) \simeq \frac{F}{2l} x^2.$$

The restoring potential therefore comes from the first-order term  $F \delta l$ , *i.e.* from the tension already present at equilibrium, and not from the stiffness  $V''(l)$  of the spring.

The kinetic energy is

$$T = \frac{1}{2} m \dot{x}^2,$$

and the equation of motion becomes

$$m \ddot{x} + \frac{F}{l} x = 0.$$

As an independent check, the tension  $V'(d)$  pulls the particle towards  $A$  along the spring, and its component along the line is

$$F_x = -V'(d) \frac{x}{d} \simeq -\frac{F}{l} x$$

to first order in  $x$ , which reproduces this equation of motion. Comparison with  $\ddot{x} + \omega^2 x = 0$  gives the angular frequency.

**ANSWER TO PROBLEM 3.**

$$\omega = \sqrt{\frac{F}{ml}}$$

## PROBLEM 4.

The same as §21 Prob. 3, but for a particle of mass  $m$  moving on a circle of radius  $r$  (Fig. 23).

SOLUTION: Let  $O$  be the centre of the circle of radius  $r$ . The fixed end  $A$  lies outside the circle, with  $OA = l + r$ , so that the spring has length  $l$  when the particle is at the point nearest  $A$ . (If  $A$  were inside the circle, with  $OA = r - l$ , the same steps would give  $\omega^2 = F(r - l)/(mrl)$ .) Let  $\phi$  be the angle between  $OA$  and the radius to the particle.

By the cosine rule, the length of the spring is

$$\begin{aligned} d(\phi) &= \sqrt{r^2 + (l + r)^2 - 2r(l + r)\cos\phi} \\ &= \sqrt{l^2 + 2r(l + r)(1 - \cos\phi)}. \end{aligned}$$

For small oscillations,  $1 - \cos\phi \simeq \phi^2/2$ . Factoring out  $l$  and using  $\sqrt{1 + \epsilon} \simeq 1 + \epsilon/2$ , we get

$$d = l\sqrt{1 + \frac{2r(l + r)}{l^2}(1 - \cos\phi)} \simeq l + \frac{r(l + r)}{2l}\phi^2,$$

and therefore

$$\delta l = d(\phi) - l \simeq \frac{r(l + r)}{2l}\phi^2.$$

The amplitude must be small enough for both expansions to hold:  $\phi \ll 1$  for the cosine, and  $r(l + r)\phi^2/l^2 \ll 1$  for the square root. As in §21 Prob. 3, the leading change in the energy of the spring is  $F\delta l$ . Hence,

$$U(\phi) \simeq \frac{Fr(l + r)}{2l}\phi^2.$$

The speed of the particle is  $r\dot{\phi}$ , giving

$$T = \frac{1}{2}mr^2\dot{\phi}^2.$$

The linearized equation of motion is

$$mr^2\ddot{\phi} + \frac{Fr(l + r)}{l}\phi = 0,$$

or

$$\ddot{\phi} + \frac{F(l + r)}{mrl}\phi = 0.$$

As a consistency check, at fixed  $l$  the circle becomes a straight line at distance  $l$  from  $A$  as  $r \rightarrow \infty$ , and

$$\omega^2 = \frac{F}{ml} \left(1 + \frac{l}{r}\right) \rightarrow \frac{F}{ml} \quad \text{as } r \rightarrow \infty,$$

which recovers the result of §21 Prob. 3.

## ANSWER TO PROBLEM 4.

$$\omega = \sqrt{\frac{F(l + r)}{mrl}}$$

## PROBLEM 5.

Find the frequency of oscillations of the pendulum shown in the figure in §5 Prob. 2, whose point of support carries a mass  $m_1$  and is free to move horizontally (its motion is integrated in §14 Prob. 3).

SOLUTION: Let  $x$  denote the horizontal position of the support and  $\phi$  the angle of the pendulum from the downward vertical. The support has mass  $m_1$ , the bob has mass  $m_2$ , and the length of the pendulum is  $l$ .

From §5 Prob. 2, the Lagrangian is

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2l^2\dot{\phi}^2 + m_2l\dot{x}\dot{\phi}\cos\phi + m_2gl\cos\phi.$$

Because  $x$  is cyclic, the total horizontal momentum is conserved:

$$p_x = \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x} + m_2l\dot{\phi}\cos\phi.$$

A constant horizontal translation of the whole system does not affect its oscillation frequency. We may therefore work in the inertial frame in which  $p_x = 0$ , as in §14 Prob. 3. In this frame,

$$\dot{x} = -\frac{m_2l}{m_1 + m_2}\dot{\phi}\cos\phi.$$

Substituting this expression into the kinetic energy, we get

$$\begin{aligned} T &= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2l^2\dot{\phi}^2 + m_2l\dot{x}\dot{\phi}\cos\phi \\ &= \frac{m_2^2l^2}{2(m_1 + m_2)}\dot{\phi}^2\cos^2\phi + \frac{1}{2}m_2l^2\dot{\phi}^2 - \frac{m_2^2l^2}{m_1 + m_2}\dot{\phi}^2\cos^2\phi \\ &= \frac{1}{2}m_2l^2\left(1 - \frac{m_2}{m_1 + m_2}\cos^2\phi\right)\dot{\phi}^2. \end{aligned}$$

For small oscillations, we retain terms of second order in the displacement and velocity. Since  $\dot{\phi}^2$  is already of second order, replacing  $\cos^2\phi$  by 1 changes  $T$  only at fourth order. Hence,

$$T \simeq \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}l^2\dot{\phi}^2.$$

Taking the gravitational potential energy to vanish at  $\phi = 0$ , we have

$$U = m_2gl(1 - \cos\phi) \simeq \frac{1}{2}m_2gl\phi^2.$$

Eliminating a cyclic velocity directly in the Lagrangian is not legitimate in general. The correct Lagrangian for the remaining co-ordinate is  $L - p_x\dot{x}$  (minus the Routhian), with  $\dot{x}$  expressed in terms of  $p_x$  and  $\phi$ . Here,  $p_x = 0$ , so the extra term vanishes and the Lagrangian for  $\phi$  is simply  $T - U$  with  $\dot{x}$  eliminated. Equivalently, the energy  $E = T + U$  then takes the one-dimensional form  $\frac{1}{2}M(\phi)\dot{\phi}^2 + U(\phi)$ , which determines the frequency.

The quadratic Lagrangian for the oscillation is therefore

$$L^{(2)} = \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}l^2\dot{\phi}^2 - \frac{1}{2}m_2gl\phi^2.$$

Its Euler–Lagrange equation gives

$$\frac{m_1m_2}{m_1 + m_2}l^2\ddot{\phi} + m_2gl\phi = 0,$$

or

$$\ddot{\phi} + \frac{g(m_1 + m_2)}{m_1l}\phi = 0.$$

The freedom of the support to recoil reduces the effective inertia of the oscillatory co-ordinate. Consequently, the frequency exceeds that of a pendulum with a fixed support. In the limit  $m_1 \rightarrow \infty$ , the

fixed-support result  $\sqrt{g/l}$  is recovered.

ANSWER TO PROBLEM 5.

$$\omega = \sqrt{\frac{g(m_1 + m_2)}{m_1 l}}$$

## PROBLEM 6.

Determine the form of a curve such that the frequency of oscillations of a particle on it under the force of gravity is independent of the amplitude.

SOLUTION: Consider a particle constrained to slide without friction along a plane curve in a uniform gravitational field. We set the lowest point at  $x = y = 0$ , with  $y$  measured vertically upwards, and let  $s$  be the signed arc length along the curve, measured from that point.

The arc length is a useful generalized co-ordinate because the speed is exactly  $|\dot{s}|$ . Thus,

$$T = \frac{1}{2}m\dot{s}^2, \quad U = mgy(s),$$

and the equation of motion is

$$\ddot{s} + g \frac{dy}{ds} = 0.$$

To construct the required curve, we require the motion in  $s$  to be exactly harmonic with a prescribed constant angular frequency  $\omega$ :

$$\ddot{s} + \omega^2 s = 0.$$

This is sufficient to make the frequency independent of the amplitude. Comparing the two equations and integrating, with  $y(0) = 0$ , gives

$$g \frac{dy}{ds} = \omega^2 s, \quad y = \frac{\omega^2 s^2}{2g}.$$

No small-amplitude expansion has been used.

The required relation between height and arc length now determines the geometry. On the right-hand branch,  $s > 0$ , we have

$$s = \frac{\sqrt{2gy}}{\omega}, \quad \left( \frac{ds}{dy} \right)^2 = \frac{g}{2\omega^2 y}.$$

Since  $ds^2 = dx^2 + dy^2$ ,

$$\left( \frac{dx}{dy} \right)^2 = \left( \frac{ds}{dy} \right)^2 - 1 = \frac{g}{2\omega^2 y} - 1.$$

We introduce the length  $a = g/(4\omega^2)$ . Then,

$$\frac{dx}{dy} = \sqrt{\frac{2a - y}{y}}.$$

We use the substitution

$$y = a(1 - \cos \theta), \quad 0 \leq \theta < \pi,$$

for which

$$dy = a \sin \theta d\theta, \quad \sqrt{\frac{2a - y}{y}} = \cot \frac{\theta}{2}.$$

Therefore,

$$dx = a \sin \theta \cot \frac{\theta}{2} d\theta = a(1 + \cos \theta) d\theta.$$

Integrating from the lowest point yields

$$x = a(\theta + \sin \theta), \quad y = a(1 - \cos \theta).$$

Reflection gives the left-hand branch, represented by the same formulae with negative  $\theta$ . This is a cycloidal trough, with its lowest point at  $\theta = 0$  and adjacent cusps at  $\theta = \pm\pi$ .

We now verify the exact independence of the amplitude explicitly. On the smooth arc  $-\pi < \theta < \pi$ ,

$$\begin{aligned}\left(\frac{ds}{d\theta}\right)^2 &= \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 \\ &= a^2(1 + \cos \theta)^2 + a^2 \sin^2 \theta \\ &= 4a^2 \cos^2 \frac{\theta}{2}.\end{aligned}$$

Choosing  $s$  to increase with  $\theta$ , we get

$$s = 4a \sin \frac{\theta}{2}.$$

It follows that

$$y = 2a \sin^2 \frac{\theta}{2} = \frac{s^2}{8a}.$$

Thus, the exact Lagrangian becomes

$$L = \frac{1}{2}m\dot{s}^2 - \frac{mg}{8a}s^2,$$

and the exact equation of motion is

$$\ddot{s} + \frac{g}{4a}s = 0.$$

Hence, every oscillation with amplitude  $|s| < 4a$ , *i.e.* whose turning points lie below the height  $2a$  of the cusps, has the same angular frequency  $\omega = \sqrt{g/4a}$  and period  $\tau = 2\pi/\omega$ . This applies to finite amplitudes, not merely to infinitesimal oscillations.

#### ANSWER TO PROBLEM 6.

The curve is a cycloidal trough, where  $a > 0$  is the radius of its generating circle, with angular frequency  $\omega$  and period  $\tau$ :

$$\begin{aligned}x &= a(\theta + \sin \theta), \\ y &= a(1 - \cos \theta), \quad -\pi < \theta < \pi, \\ \omega &= \sqrt{\frac{g}{4a}}, \quad \tau = 4\pi\sqrt{\frac{a}{g}}.\end{aligned}$$

**§22 FORCED OSCILLATIONS**

PROBLEM 1.

*Not done*

Determine the forced oscillations of a system under a force  $F(t)$  of the following forms, if at time  $t = 0$  the system is at rest in equilibrium ( $x = dx/dt = 0$ ):

(a)

*Not done* $F = F_0$ , a constant.

(b)

*Not done* $F = at$ .

(c)

*Not done* $F = F_0 e^{-\alpha t}$ .

(d)

*Not done* $F = F_0 e^{-\alpha t} \cos \beta t$ .

## PROBLEM 2.

*Not done*

Determine the final amplitude for the oscillations of a system under a force which is zero for  $t < 0$ ,  $F_0 t/T$  for  $0 < t < T$ , and  $F_0$  for  $t > T$  (Fig. 24), if up to time  $t = 0$  the system is at rest in equilibrium.

## PROBLEM 3.

*Not done*

The same as §22 Prob. 2, but for a constant force  $F_0$  which acts for a finite time  $T$  (Fig. 25).

## PROBLEM 4.

*Not done*

The same as §22 Prob. 2, but for a force  $F_0 t/T$  which acts between  $t = 0$  and  $t = T$  (Fig. 26).

## PROBLEM 5.

*Not done*

The same as §22 Prob. 2, but for a force  $F_0 \sin \omega t$  which acts between  $t = 0$  and  $t = T = 2\pi/\omega$  (Fig. 27).

**§23** OSCILLATIONS OF SYSTEMS WITH MORE THAN ONE DEGREE OF FREEDOM

PROBLEM 1.

*Not done*

Determine the oscillations of a system with two degrees of freedom whose Lagrangian is

$$L = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) - \frac{1}{2} \omega_0^2 (x^2 + y^2) + \alpha xy$$

(two identical one-dimensional systems of eigenfrequency  $\omega_0$  coupled by an interaction  $-\alpha xy$ ).

## PROBLEM 2.

*Not done*

Determine the small oscillations of a coplanar double pendulum (see the figure in §5 Prob. 1).

## PROBLEM 3.

*Not done*

Find the path of a particle in a central field  $U = \frac{1}{2}kr^2$  (called a space oscillator).

**§24** VIBRATIONS OF MOLECULES

PROBLEM 1.

*Not done*

Determine the frequencies of vibrations of a symmetrical linear triatomic molecule  $ABA$  (Fig. 28). It is assumed that the potential energy of the molecule depends only on the distances  $AB$  and  $BA$  and the angle  $ABA$ .

## PROBLEM 2.

*Not done*

The same as §24 Prob. 1, but for a triangular molecule  $ABA$  (Fig. 29).

## PROBLEM 3.

*Not done*

The same as §24 Prob. 1, but for an unsymmetrical linear molecule  $ABC$  (Fig. 30).

**§26** FORCED OSCILLATIONS UNDER FRICTION

PROBLEM 1.

*Not done*

Determine the forced oscillations due to an external force  $f = f_0 e^{\alpha t} \cos \gamma t$  in the presence of friction.

**§27** PARAMETRIC RESONANCE

PROBLEM 1.

*Not done*

Obtain an expression correct as far as the term in  $h^2$  for the limits of the region of instability for resonance near  $\gamma = 2\omega_0$ .

## PROBLEM 2.

*Not done*

Determine the limits of the region of instability in resonance near  $\gamma = \omega_0$ .

## PROBLEM 3.

*Not done*

Find the conditions for parametric resonance in small oscillations of a simple pendulum whose point of support oscillates vertically (see the Lagrangian in §5 Prob. 3(c)).

**§29** RESONANCE IN NON-LINEAR OSCILLATIONS

PROBLEM 1.

*Not done*

Determine the function  $b(\epsilon)$  for resonance at frequencies  $\gamma \approx 3\omega_0$ .

**§30** MOTION IN A RAPIDLY OSCILLATING FIELD

PROBLEM 1.

*Not done*

Determine the positions of stable equilibrium of a pendulum whose point of support oscillates vertically with a high frequency  $\gamma \gg \sqrt{g/l}$  (see the Lagrangian in §5 Prob. 3(c)).

## PROBLEM 2.

*Not done*

The same as §30 Prob. 1, but for a pendulum whose point of support oscillates horizontally (see the Lagrangian in §5 Prob. 3(b)).

## CHAPTER 6.

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### MOTION OF A RIGID BODY

## §32 THE INERTIA TENSOR

## PROBLEM 1.

Determine the principal moments of inertia for the following types of molecule, regarded as systems of particles at fixed distances apart:

(a)

A molecule of collinear atoms.

SOLUTION: Choose the  $x_3$ -axis along the line containing all atoms, and place the origin at the centre of mass. Then each particle has coordinates  $(0, 0, x_a)$ , so the inertia tensor is diagonal in the basis  $(x_1, x_2, x_3)$ , with

$$I_3 = 0, \quad I_1 = I_2.$$

Indeed, since the distance to the  $x_1$ -axis (or  $x_2$ -axis) is  $|x_a - x_{CM}|$  for each particle,

$$I_1 = I_2 = \sum_a m_a (x_a - x_{CM})^2, \quad x_{CM} = \frac{1}{\mu} \sum_a m_a x_a, \quad \mu = \sum_a m_a.$$

Expanding and using the definition of  $x_{CM}$  gives

$$\begin{aligned} I_1 = I_2 &= \sum_a m_a x_a^2 - 2x_{CM} \sum_a m_a x_a + \mu x_{CM}^2 \\ &= \sum_a m_a x_a^2 - \frac{1}{\mu} \left( \sum_a m_a x_a \right)^2. \end{aligned}$$

To rewrite this in terms of pairwise separations, expand the square:

$$\left( \sum_a m_a x_a \right)^2 = \sum_a m_a^2 x_a^2 + 2 \sum_{a < b} m_a m_b x_a x_b.$$

Hence

$$\begin{aligned} I_1 = I_2 &= \sum_a m_a x_a^2 - \frac{1}{\mu} \sum_a m_a^2 x_a^2 - \frac{2}{\mu} \sum_{a < b} m_a m_b x_a x_b \\ &= \frac{1}{\mu} \sum_a m_a (\mu - m_a) x_a^2 - \frac{2}{\mu} \sum_{a < b} m_a m_b x_a x_b. \end{aligned}$$

Now note that  $\mu - m_a = \sum_{b \neq a} m_b$ , so

$$\sum_a m_a (\mu - m_a) x_a^2 = \sum_a \sum_{b \neq a} m_a m_b x_a^2 = \sum_{a < b} m_a m_b (x_a^2 + x_b^2).$$

Therefore

$$\begin{aligned} I_1 = I_2 &= \frac{1}{\mu} \sum_{a < b} m_a m_b (x_a^2 + x_b^2) - \frac{2}{\mu} \sum_{a < b} m_a m_b x_a x_b \\ &= \frac{1}{\mu} \sum_{a < b} m_a m_b (x_a - x_b)^2. \end{aligned}$$

Since for collinear atoms the pair distance is  $l_{ab} = |x_a - x_b|$ , we finally obtain

ANSWER TO PROBLEM 1 (A).

$$\begin{aligned} I_1 = I_2 &= \frac{1}{\mu} \sum_{a < b} m_a m_b l_{ab}^2 \\ I_3 &= 0 \end{aligned}$$

(b)

A triatomic molecule which is an isosceles triangle (Fig. 36).

SOLUTION: Let the molecule lie in the  $(x_1, x_2)$ -plane, with  $x_1$  along the base and  $x_2$  along the axis of symmetry. The two base atoms have equal masses  $m_1$  and are separated by a distance  $a$ ; the apex atom has mass  $m_2$  and is at height  $h$  above the base midpoint. Denote the total mass by

$$\mu = 2m_1 + m_2.$$

Choose coordinates (before shifting to the centre of mass):

$$(x_1, x_2)_{(1)} = \left(-\frac{a}{2}, 0\right), \quad (x_1, x_2)_{(3)} = \left(+\frac{a}{2}, 0\right), \quad (x_1, x_2)_{(2)} = (0, h),$$

where atoms (1) and (3) have mass  $m_1$  and atom (2) has mass  $m_2$ . By symmetry  $x_{1,CM} = 0$ , and

$$x_{2,CM} = \frac{m_2 h}{\mu}.$$

The symmetry implies that the axes through the centre of mass parallel to  $x_1$ ,  $x_2$ , and  $x_3$  are principal. Hence,

$$I_1 = \sum_a m_a (x_{2,a} - x_{2,CM})^2, \quad I_2 = \sum_a m_a (x_{1,a} - x_{1,CM})^2, \quad I_3 = I_1 + I_2,$$

since the molecule is planar ( $x_3 \equiv 0$ ). Using the pair-distance identity in one dimension,

$$\sum_a m_a (x_{2,a} - x_{2,CM})^2 = \frac{1}{\mu} \sum_{a < b} m_a m_b (x_{2,a} - x_{2,b})^2,$$

and noting that the only non-zero vertical separations are between the apex and each base atom, equal to  $h$ , we obtain

$$I_1 = \frac{1}{\mu} (m_1 m_2 h^2 + m_1 m_2 h^2) = \frac{2m_1 m_2}{\mu} h^2.$$

Similarly,

$$I_2 = \sum_a m_a (x_{1,a} - x_{1,CM})^2 = \frac{1}{\mu} \sum_{a < b} m_a m_b (x_{1,a} - x_{1,b})^2.$$

The horizontal separations are between the two base atoms, and between the apex and either base atom. Hence

$$I_2 = \frac{1}{\mu} \left( m_1^2 a^2 + 2m_1 m_2 \left(\frac{a}{2}\right)^2 \right) = \frac{1}{\mu} \left( m_1^2 a^2 + \frac{1}{2} m_1 m_2 a^2 \right) = \frac{1}{2} m_1 a^2.$$

Therefore

$$I_3 = I_1 + I_2 = \frac{2m_1 m_2}{\mu} h^2 + \frac{1}{2} m_1 a^2.$$

ANSWER TO PROBLEM 1 (B).

$$I_1 = \frac{2m_1 m_2}{\mu} h^2$$

$$I_2 = \frac{1}{2} m_1 a^2$$

$$I_3 = I_1 + I_2.$$

(c)

A tetratomic molecule which is an equilateral-based tetrahedron (Fig. 37).

SOLUTION: The base is an equilateral triangle of side  $a$  in the plane  $x_3 = 0$ , with three equal masses  $m_1$  at its vertices. A fourth atom of mass  $m_2$  lies on the symmetry axis through the base centroid at height  $h$  above the base. Let

$$\mu = 3m_1 + m_2.$$

Let  $\rho$  be the distance from the base centroid to a base vertex. For an equilateral triangle,

$$\rho = \frac{a}{\sqrt{3}}, \quad \rho^2 = \frac{a^2}{3}.$$

By symmetry the centre of mass lies on the symmetry axis. Measuring  $x_3$  from the base plane,

$$x_{3,CM} = \frac{m_2 h}{\mu}.$$

The symmetry axis is a principal axis; any two orthogonal axes through the centre of mass perpendicular to it are also principal and have equal moments. Thus

$$I_1 = I_2, \quad I_3 \text{ about the symmetry axis } (x_3).$$

The apex mass lies on the axis and contributes nothing. Each base mass is at distance  $\rho$  from the axis, hence

$$I_3 = 3m_1 \rho^2 = 3m_1 \frac{a^2}{3} = m_1 a^2.$$

Let  $x_1$  be any axis through the centre of mass perpendicular to  $x_3$ . Then

$$I_1 = \sum_a m_a (y_a^2 + z_a^2),$$

where  $y$  is a coordinate orthogonal to  $x_1$  in the plane perpendicular to the axis, and  $z$  is measured along  $x_3$  from the centre of mass. The vertical coordinates relative to the centre of mass are

$$z_{\text{base}} = -x_{3,CM} = -\frac{m_2 h}{\mu}, \quad z_{\text{apex}} = h - x_{3,CM} = h \left(1 - \frac{m_2}{\mu}\right) = \frac{3m_1}{\mu} h.$$

For the three vertices of an equilateral triangle, for any in-plane axis through the centroid,

$$\sum_{\text{base}} y^2 = \frac{3}{2} \rho^2 = \frac{3}{2} \cdot \frac{a^2}{3} = \frac{1}{2} a^2.$$

Therefore,

$$\begin{aligned} I_1 &= m_1 \left( \sum_{\text{base}} y^2 \right) + 3m_1 z_{\text{base}}^2 + m_2 z_{\text{apex}}^2 \\ &= \frac{1}{2} m_1 a^2 + 3m_1 \left( \frac{m_2^2 h^2}{\mu^2} \right) + m_2 \left( \frac{9m_1^2 h^2}{\mu^2} \right) \\ &= \frac{1}{2} m_1 a^2 + \frac{3m_1 m_2}{\mu} h^2. \end{aligned}$$

Hence  $I_2 = I_1$ .

ANSWER TO PROBLEM 1 (C).

$$\begin{aligned} I_1 = I_2 &= \frac{3m_1 m_2}{\mu} h^2 + \frac{1}{2} m_1 a^2 \\ I_3 &= m_1 a^2 \end{aligned}$$

## PROBLEM 2.

*Not done*

Determine the principal moments of inertia for the following homogeneous bodies:

(a)

*Not done*

A thin rod of length  $l$ .

(b)

*Not done*

A sphere of radius  $R$ .

(c)

*Not done*

A circular cylinder of radius  $R$  and height  $h$ .

(d)

*Not done*

A rectangular parallelepiped of sides  $a$ ,  $b$ , and  $c$ .

(e)

*Not done*

A circular cone of height  $h$  and base radius  $R$ .

(f)

*Not done*

An ellipsoid of semiaxes  $a$ ,  $b$ ,  $c$ .

## PROBLEM 3.

*Not done*

Determine the frequency of small oscillations of a compound pendulum (a rigid body swinging about a fixed horizontal axis in a gravitational field).

## PROBLEM 4.

*Not done*

Find the kinetic energy of the system shown in Fig. 39:  $OA$  and  $AB$  are thin uniform rods of length  $l$  hinged together at  $A$ . The rod  $OA$  rotates (in the plane of the diagram) about  $O$ , while the end  $B$  of the rod  $AB$  slides along  $Ox$ .

## PROBLEM 5.

*Not done*

Find the kinetic energy of a cylinder of radius  $R$  rolling on a plane, if the mass of the cylinder is so distributed that one of the principal axes of inertia is parallel to the axis of the cylinder and at a distance  $a$  from it, and the moment of inertia about that principal axis is  $I$ .

## PROBLEM 6.

*Not done*

Find the kinetic energy of a homogeneous cylinder of radius  $a$  rolling inside a cylindrical surface of radius  $R$  (Fig. 41).

## PROBLEM 7.

*Not done*

Find the kinetic energy of a homogeneous cone rolling on a plane (its principal moments of inertia are found in §32 Prob. 2(e)).

## PROBLEM 8.

*Not done*

Find the kinetic energy of a homogeneous cone whose base rolls on a plane and whose vertex is fixed at a height above the plane equal to the radius of the base, so that the axis of the cone is parallel to the plane (see §32 Prob. 2(e) and §32 Prob. 7).

## PROBLEM 9.

*Not done*

Find the kinetic energy of a homogeneous ellipsoid which rotates about one of its axes ( $AB$  in Fig. 44) while that axis itself rotates about a line  $CD$  perpendicular to it and passing through the centre of the ellipsoid.

## PROBLEM 10.

*Not done*

The same as §32 Prob. 9, but for the case where the axis  $AB$  is not perpendicular to  $CD$  and is an axis of symmetry of the ellipsoid (Fig. 45).

**§35** EULERIAN ANGLES

PROBLEM 1.

*Not done*

Reduce to quadratures the problem of the motion of a heavy symmetrical top whose lowest point is fixed (Fig. 48).

## PROBLEM 2.

*Not done*

Find the condition for the rotation of a top about a vertical axis to be stable (see §35 Prob. 1).

## PROBLEM 3.

*Not done*

Determine the motion of a top when the kinetic energy of its rotation about its axis is large compared with its energy in the gravitational field (called a “fast” top; see §35 Prob. 1).

**§37** THE ASYMMETRICAL TOP

PROBLEM 1.

*Not done*

Determine the free rotation of a top about an axis near the  $x_3$ -axis or the  $x_1$ -axis.

## PROBLEM 2.

*Not done*

Determine the free rotation of a top for which  $M^2 = 2EI_2$ .

**§38** RIGID BODIES IN CONTACT

PROBLEM 1.

*Not done*

Using d'Alembert's principle, find the equations of motion of a homogeneous sphere rolling on a plane under an external force  $\mathbf{F}$  and torque  $\mathbf{K}$ .

## PROBLEM 2.

*Not done*

A uniform rod  $BD$  of weight  $P$  and length  $l$  rests against a wall as shown in Fig. 52 and its lower end  $B$  is held by a string  $AB$ . Find the reaction of the wall and the tension in the string.

## PROBLEM 3.

*Not done*

A rod of weight  $P$  has one end  $A$  on a vertical plane and the other end  $B$  on a horizontal plane (Fig. 53), and is held in position by two horizontal strings  $AD$  and  $BC$ , the latter being in the same vertical plane as  $AB$ . Determine the reactions of the planes and the tensions in the strings.

## PROBLEM 4.

*Not done*

Two rods of length  $l$  and negligible weight are hinged together, and their ends are connected by a string  $AB$  (Fig. 54). They stand on a plane, and a force  $F$  is applied at the midpoint of one rod. Determine the reactions.

**§39** MOTION IN A NON-INERTIAL FRAME OF REFERENCE

PROBLEM 1.

*Not done*

Find the deflection of a freely falling body from the vertical caused by the Earth's rotation, assuming the angular velocity of this rotation to be small.

## PROBLEM 2.

*Not done*

Determine the deflection from coplanarity of the path of a particle thrown from the Earth's surface with velocity  $\mathbf{v}_0$ .

## PROBLEM 3.

*Not done*

Determine the effect of the Earth's rotation on small oscillations of a pendulum (the problem of Foucault's pendulum).

## CHAPTER 7.

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### THE CANONICAL EQUATIONS

## §40 HAMILTON'S EQUATIONS

## PROBLEM 1.

Find the Hamiltonian for a single particle in Cartesian, cylindrical and spherical co-ordinates.

SOLUTION: For a particle with a velocity-independent potential energy  $U$ , the generalized momenta and the Hamiltonian are

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad H = \sum_i p_i \dot{q}_i - L.$$

The kinetic energy is quadratic in the velocities, so the Legendre transform gives  $H = T + U$ , with the velocities expressed in terms of the canonical momenta.

In Cartesian co-ordinates, we have

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z), \quad p_x = m\dot{x}, \quad p_y = m\dot{y}, \quad p_z = m\dot{z}.$$

In cylindrical co-ordinates, the Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) - U(r, \phi, z),$$

so that

$$p_r = m\dot{r}, \quad p_\phi = mr^2\dot{\phi}, \quad p_z = m\dot{z}.$$

In spherical co-ordinates, with  $\theta$  the polar angle and  $\phi$  the azimuth, the Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2 \theta) - U(r, \theta, \phi),$$

and hence

$$p_r = m\dot{r}, \quad p_\theta = mr^2\dot{\theta}, \quad p_\phi = mr^2\dot{\phi} \sin^2 \theta.$$

Substituting these momenta in  $H = T + U$  gives the three Hamiltonians.

## ANSWER TO PROBLEM 1.

$$\text{Cartesian: } H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + U(x, y, z)$$

$$\text{Cylindrical: } H = \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2} + p_z^2\right) + U(r, \phi, z)$$

$$\text{Spherical: } H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta}\right) + U(r, \theta, \phi)$$

## PROBLEM 2.

Find the Hamiltonian for a particle in a uniformly rotating frame of reference.

SOLUTION: Let  $\mathbf{v} = \dot{\mathbf{r}}$  be the velocity relative to a frame rotating with constant angular velocity  $\boldsymbol{\Omega}$  about a fixed origin. The inertial velocity, expressed along the rotating axes, is  $\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}$ . Thus, the Lagrangian is

$$L = \frac{1}{2}m(\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r})^2 - U,$$

and the canonical momentum is

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = m(\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}).$$

Consequently,

$$\mathbf{v} = \frac{\mathbf{p}}{m} - \boldsymbol{\Omega} \times \mathbf{r}.$$

The Legendre transform gives

$$\begin{aligned} H &= \mathbf{p} \cdot \mathbf{v} - L \\ &= \mathbf{p} \cdot \left( \frac{\mathbf{p}}{m} - \boldsymbol{\Omega} \times \mathbf{r} \right) - \frac{\mathbf{p}^2}{2m} + U \\ &= \frac{\mathbf{p}^2}{2m} - \boldsymbol{\Omega} \cdot (\mathbf{r} \times \mathbf{p}) + U, \end{aligned}$$

where we used  $\mathbf{p} \cdot (\boldsymbol{\Omega} \times \mathbf{r}) = \boldsymbol{\Omega} \cdot (\mathbf{r} \times \mathbf{p})$ . Here,  $\mathbf{p}$  is the canonical momentum, rather than  $m\mathbf{v}$ .

## ANSWER TO PROBLEM 2.

$$H = \frac{\mathbf{p}^2}{2m} - \boldsymbol{\Omega} \cdot (\mathbf{r} \times \mathbf{p}) + U$$

## PROBLEM 3.

Find the Hamiltonian for a system comprising one particle of mass  $M$  and  $n$  particles each of mass  $m$ , excluding the motion of the centre of mass (see §13 Prob. 1).

SOLUTION: We use the relative co-ordinates of §13 Prob. 1, namely  $\mathbf{r}_a = \mathbf{R}_a - \mathbf{R}$  and  $\mathbf{v}_a = \dot{\mathbf{r}}_a$ , with  $\mu \equiv M + nm$  the total mass. The reduced Lagrangian, including the potential energy, is

$$L = \frac{1}{2}m \sum_a \mathbf{v}_a^2 - \frac{1}{2} \frac{m^2}{\mu} \left( \sum_a \mathbf{v}_a \right)^2 - U.$$

The canonical momenta are

$$\mathbf{p}_a = \frac{\partial L}{\partial \mathbf{v}_a} = m\mathbf{v}_a - \frac{m^2}{\mu} \sum_b \mathbf{v}_b.$$

Summing over  $a$ , we get

$$\sum_a \mathbf{p}_a = \left( m - \frac{nm^2}{\mu} \right) \sum_a \mathbf{v}_a = \frac{mM}{\mu} \sum_a \mathbf{v}_a.$$

Substituting this relation back in the expression for  $\mathbf{p}_a$  yields

$$\mathbf{v}_a = \frac{\mathbf{p}_a}{m} + \frac{1}{M} \sum_b \mathbf{p}_b.$$

Since the reduced kinetic energy is quadratic in the velocities, we have  $\sum_a \mathbf{p}_a \cdot \mathbf{v}_a = 2T$  and

$$\begin{aligned} H = T + U &= \frac{1}{2} \sum_a \mathbf{p}_a \cdot \mathbf{v}_a + U \\ &= \frac{1}{2m} \sum_a \mathbf{p}_a^2 + \frac{1}{2M} \left( \sum_a \mathbf{p}_a \right)^2 + U. \end{aligned}$$

The momenta  $\mathbf{p}_a$  are conjugate to the relative co-ordinates  $\mathbf{r}_a$ , and the kinetic energy of the centre of mass has already been removed.

## ANSWER TO PROBLEM 3.

$$H = \frac{1}{2m} \sum_a \mathbf{p}_a^2 + \frac{1}{2M} \left( \sum_a \mathbf{p}_a \right)^2 + U(\mathbf{r}_1, \dots, \mathbf{r}_n)$$

## §41 THE ROUTHIAN

## PROBLEM 1.

Find the Routhian for a symmetrical top in an external field  $U(\phi, \theta)$ , eliminating the cyclic coordinate  $\psi$  (where  $\psi, \phi, \theta$  are Eulerian angles).

SOLUTION: Let  $I'_1 = I'_2$  be the transverse moments of inertia about the fixed point and  $I_3$  the moment of inertia about the axis of symmetry. In Eulerian angles, the Lagrangian is

$$L = \frac{I'_1}{2} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{I_3}{2} (\dot{\psi} + \dot{\phi} \cos \theta)^2 - U(\phi, \theta).$$

Since  $L$  is independent of  $\psi$ , its conjugate momentum is conserved:

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3 (\dot{\psi} + \dot{\phi} \cos \theta).$$

Solving for the velocity to be eliminated gives

$$\dot{\psi} = \frac{p_\psi}{I_3} - \dot{\phi} \cos \theta.$$

The Routhian is the partial Legendre transform  $R = p_\psi \dot{\psi} - L$ . Substituting the expression for  $\dot{\psi}$ , we get

$$R = p_\psi \left( \frac{p_\psi}{I_3} - \dot{\phi} \cos \theta \right) - \frac{I'_1}{2} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) - \frac{p_\psi^2}{2I_3} + U(\phi, \theta).$$

Thus,  $\psi$  and  $\dot{\psi}$  are eliminated, while  $p_\psi$  is retained as a constant parameter.

## ANSWER TO PROBLEM 1.

$$R(\phi, \theta, \dot{\phi}, \dot{\theta}; p_\psi) = \frac{p_\psi^2}{2I_3} - p_\psi \dot{\phi} \cos \theta - \frac{I'_1}{2} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + U(\phi, \theta)$$

For a given value of  $p_\psi$ , the term  $p_\psi^2/2I_3$  is a constant, so it may be omitted when deriving the reduced equations of motion for  $\phi$  and  $\theta$ .

## §42 POISSON BRACKETS

## PROBLEM 1.

Determine the Poisson brackets formed from the Cartesian components of the momentum  $\mathbf{p}$  and the angular momentum  $\mathbf{M} = \mathbf{r} \times \mathbf{p}$  of a particle.

SOLUTION: We use Landau's definition of the Poisson bracket,

$$[F, G] = \sum_{k=x,y,z} \left( \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial x_k} - \frac{\partial F}{\partial x_k} \frac{\partial G}{\partial p_k} \right).$$

Since the momentum components are independent canonical variables, we have

$$[p_i, p_j] = 0.$$

For a bracket involving one momentum component, the definition reduces to

$$[F, p_j] = -\frac{\partial F}{\partial x_j}.$$

Using  $M_x = yp_z - zp_y$ , we obtain

$$[M_x, p_x] = 0, \quad [M_x, p_y] = -p_z, \quad [M_x, p_z] = p_y.$$

The remaining brackets follow by cyclically permuting  $x, y, z$ . In index notation, with  $\epsilon_{xyz} = 1$ , these results take the form given below. Brackets with the arguments reversed follow from antisymmetry.

## ANSWER TO PROBLEM 1.

$$[p_i, p_j] = 0, \quad [M_i, p_j] = -\sum_k \epsilon_{ijk} p_k$$

## PROBLEM 2.

Determine the Poisson brackets formed from the components of  $\mathbf{M}$ .

SOLUTION: The components of the angular momentum are

$$M_x = yp_z - zp_y, \quad M_y = zp_x - xp_z, \quad M_z = xp_y - yp_x.$$

For  $M_x$  and  $M_y$ , only the pair  $z, p_z$  contributes to the bracket defined in §42 Prob. 1:

$$[M_x, M_y] = \frac{\partial M_x}{\partial p_z} \frac{\partial M_y}{\partial z} - \frac{\partial M_x}{\partial z} \frac{\partial M_y}{\partial p_z} = yp_x - (-p_y)(-x) = yp_x - xp_y = -M_z.$$

Cyclic permutation gives the other two independent brackets. Brackets of a component with itself vanish by antisymmetry.

## ANSWER TO PROBLEM 2.

$$[M_x, M_y] = -M_z, \quad [M_y, M_z] = -M_x, \quad [M_z, M_x] = -M_y$$

or, equivalently,

$$[M_i, M_j] = - \sum_k \epsilon_{ijk} M_k$$

## PROBLEM 3.

Show that  $[\phi, M_z] = 0$ , where  $\phi$  is any function, spherically symmetrical about the origin, of the co-ordinates and momentum of a particle.

SOLUTION: Since  $M_z = xp_y - yp_x$ , the bracket of §42 Prob. 1 with any smooth function  $F(\mathbf{r}, \mathbf{p})$  is

$$[F, M_z] = y \frac{\partial F}{\partial x} - x \frac{\partial F}{\partial y} + p_y \frac{\partial F}{\partial p_x} - p_x \frac{\partial F}{\partial p_y}.$$

We denote this differential operator by  $\mathcal{D}_z F$ .

A scalar invariant under simultaneous rotations of  $\mathbf{r}$  and  $\mathbf{p}$  can depend on these vectors only through

$$u = \mathbf{r}^2, \quad v = \mathbf{p}^2, \quad w = \mathbf{r} \cdot \mathbf{p}.$$

Applying the operator to these combinations gives

$$\mathcal{D}_z u = 2xy - 2xy = 0, \quad \mathcal{D}_z v = 2p_x p_y - 2p_x p_y = 0, \quad \mathcal{D}_z w = yp_x - xp_y + xp_y - yp_x = 0.$$

For  $\phi = \Phi(u, v, w)$ , the chain rule therefore yields

$$[\phi, M_z] = \frac{\partial \Phi}{\partial u} \mathcal{D}_z u + \frac{\partial \Phi}{\partial v} \mathcal{D}_z v + \frac{\partial \Phi}{\partial w} \mathcal{D}_z w = 0.$$

## ANSWER TO PROBLEM 3.

$$[\phi, M_z] = 0$$

## PROBLEM 4.

Show that  $[\mathbf{f}, M_z] = \mathbf{f} \times \mathbf{n}$ , where  $\mathbf{f}$  is a vector function of the co-ordinates and momentum of a particle, and  $\mathbf{n}$  is a unit vector parallel to the  $z$ -axis.

SOLUTION: Here, a vector function means a function that transforms as a vector under simultaneous rotations of its arguments.

Under an infinitesimal active rotation through an angle  $\delta\alpha$  about the  $z$ -axis, we have

$$\delta\mathbf{r} = \delta\alpha \mathbf{n} \times \mathbf{r}, \quad \delta\mathbf{p} = \delta\alpha \mathbf{n} \times \mathbf{p}.$$

For each component  $f_i$ , the chain rule gives

$$\frac{\delta f_i}{\delta\alpha} = (\mathbf{n} \times \mathbf{r}) \cdot \nabla_{\mathbf{r}} f_i + (\mathbf{n} \times \mathbf{p}) \cdot \nabla_{\mathbf{p}} f_i = -y \frac{\partial f_i}{\partial x} + x \frac{\partial f_i}{\partial y} - p_y \frac{\partial f_i}{\partial p_x} + p_x \frac{\partial f_i}{\partial p_y} = -[f_i, M_z],$$

using the operator derived in §42 Prob. 3. Because  $\mathbf{f}$  transforms as a vector, its change under the same rotation is

$$\frac{\delta \mathbf{f}}{\delta\alpha} = \mathbf{n} \times \mathbf{f}.$$

Comparing these expressions component by component gives

$$[\mathbf{f}, M_z] = -\mathbf{n} \times \mathbf{f} = \mathbf{f} \times \mathbf{n}.$$

## ANSWER TO PROBLEM 4.

$$[\mathbf{f}, M_z] = \mathbf{f} \times \mathbf{n}$$

**§44** MAUPERTUIS' PRINCIPLE

PROBLEM 1.

*Not done*

Derive the differential equation of the path from the variational principle (44.10).

**§48** SEPARATION OF THE VARIABLES

PROBLEM 1.

*Not done*

Find a complete integral of the Hamilton–Jacobi equation for motion of a particle in a field  $U = \alpha/r - Fz$  (a combination of a Coulomb field and a uniform field), and find a conserved function of the co-ordinates and momenta that is specific to this motion.

## PROBLEM 2.

*Not done*

The same as §48 Prob. 1, but for a field  $U = \alpha_1/r_1 + \alpha_2/r_2$  (the Coulomb field of two fixed points at a distance  $2\sigma$  apart).

**§50** CANONICAL VARIABLES

PROBLEM 1.

*Not done*

Write down the equations of motion in canonical variables for a harmonic oscillator (whose Hamiltonian is (49.11)) with time-dependent frequency.

**§51** ACCURACY OF CONSERVATION OF THE ADIABATIC INVARIANT

PROBLEM 1.

*Not done*

Estimate  $\delta I$  for a harmonic oscillator with a frequency that varies slowly according to

$$\omega^2 = \omega_0^2 \frac{1 + ae^{\alpha t}}{1 + e^{\alpha t}}$$

from  $\omega_- = \omega_0$  for  $t = -\infty$  to  $\omega_+ = \sqrt{a}\omega_0$  for  $t = +\infty$  ( $a > 0$ ,  $\alpha \ll \omega_0$ ).

## PROBLEM 2.

*Not done*

A particle oscillates in a potential well. Determine how its energy varies under a frictional force  $f_{\text{fr}} = -\alpha\dot{x}$  with a small coefficient  $\alpha$  ( $x$  being a Cartesian co-ordinate).

**§52** CONDITIONALLY PERIODIC MOTION

PROBLEM 1.

*Not done*

Calculate the action variables for elliptic motion in a field  $U = -\alpha/r$ .